Transition to College Math

Name ______________________________

Date ____________ Period _________

Unit: **4: Trigonometric Graphs and Identities** Lesson **4: Sum and Difference Identities**	Essential Question: **Why is a phase shift of π radians equivalent to reflecting sin x about the x-axis?**
Standard: F-TF.9	Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems.
***Learning Target*:**	80% of the students will be able to determine the exact value of $\cos(75°)$ without using a calculator.
***Sum and Difference Identities*:**	Phase shifts, such as $\sin(\alpha+\beta)$ are often evaluated by using the sum and difference identities:

Sum Identities
$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$
$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$
$\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

Summary

Difference Identities

$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$
$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$
$\tan(\alpha-\beta)=\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$

***Example 1*:**

Find the exact value of each expression.

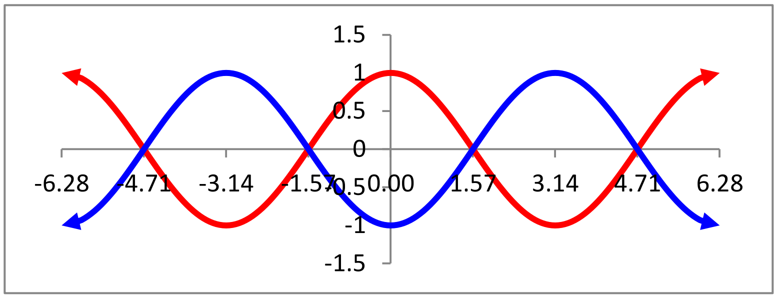
a. $\sin 75°$

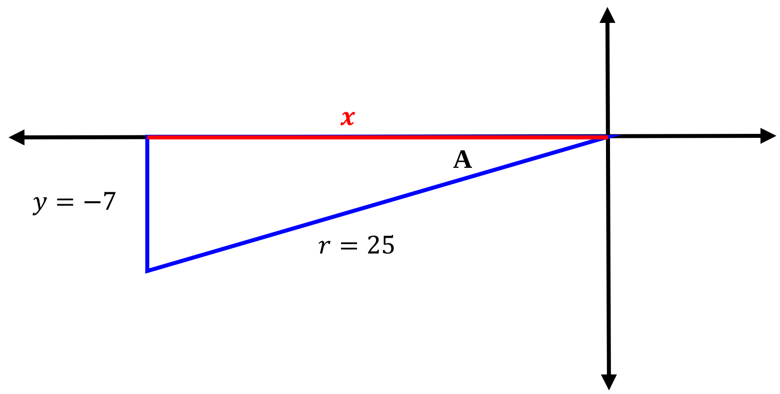
$$\sin 75° = \sin(\mathbf{30°}+\mathbf{45°})$$
$$= \sin\mathbf{30°}\cos\mathbf{45°}+\cos\mathbf{30°}\sin\mathbf{45°}$$
$$= \frac{1}{2}\cdot\frac{\sqrt{2}}{2}+\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}$$
$$= \frac{\sqrt{2}}{4}+\frac{\sqrt{6}}{4}=\frac{\sqrt{2}+\sqrt{6}}{4}$$

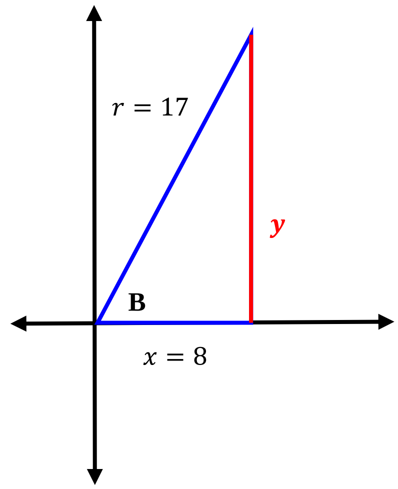
b. $\cos\left(-\frac{\pi}{12}\right)$

$$\cos\left(-\frac{\pi}{12}\right)=\cos\left(\frac{\boldsymbol{\pi}}{\mathbf{6}}-\frac{\boldsymbol{\pi}}{\mathbf{4}}\right)$$
$$= \cos\frac{\boldsymbol{\pi}}{\mathbf{6}}\cos\frac{\boldsymbol{\pi}}{\mathbf{4}}+\sin\frac{\boldsymbol{\pi}}{\mathbf{6}}\sin\frac{\boldsymbol{\pi}}{\mathbf{4}}$$
$$= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}+\frac{1}{2}\cdot\frac{\sqrt{2}}{2}$$
$$= \frac{\sqrt{6}}{4}+\frac{\sqrt{2}}{4}=\frac{\sqrt{2}+\sqrt{6}}{4}$$

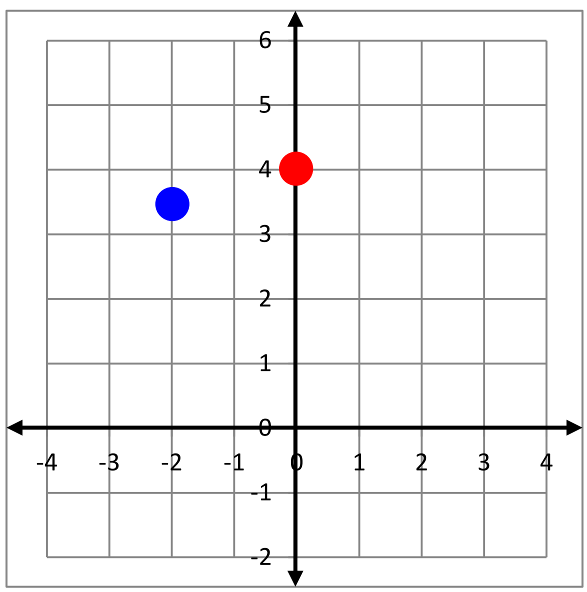
***Exercise 1*:**	Find the exact value of each expression. a. $\tan 105°$ b. $\sin\left(-\frac{11\pi}{12}\right)$

***Example 2*:**	Shifting the cosine function to the right is equivalent to reflecting it across the *x*–axis. $\boldsymbol{y = \cos x}$ $\quad$ $\boldsymbol{y = \cos(x - \pi)}$ 
***Proof*:**	$$\cos(x - \pi) = \cos x \cos \pi + \sin x \sin \pi$$ $$= \cos x \cdot (-1) + \sin x \cdot 0$$ $$= -\cos x$$ Q.E.D.
***Exercise 2*:**	Prove the identity $$\cos\left(x + \frac{\pi}{2}\right)$$

Example 3:	Find $\tan(A+B)$ given the following: $\sin A = -\frac{7}{25}$ $180° < A < 270°$ $\cos B = \frac{8}{17}$ $0° < B < 180°$
Step 1:	We must first find $\tan A$ and $\tan B$, which means finding $\cos A$ and $\sin B$. These conditions tell us that the terminal side of $\angle A$ must be in the third quadrant, and the terminal side of $\angle B$ must be in the first quadrant. Let's draw a triangle that lies in the third quadrant with its angle being the reference angle of $\angle A$.  Using the Pythagorean Theorem, $x^2 + (-7)^2 = 25^2$ $x = \pm\sqrt{625-49} = \pm 24$ Since the triangle is in the third quadrant, $x = -24$ Therefore, $\tan A = \frac{y}{x} = \frac{7}{24}$.

	Now, let's draw a triangle in quadrant I with its angle being $\angle B$.  Using the Pythagorean Theorem, $8^2 + y^2 = 17^2$ $y = \pm\sqrt{289 - 64} = \pm 15$ Since the triangle is in the first quadrant, $y = 15$ Therefore, $\tan B = \frac{y}{x} = \frac{15}{8}.$
***Step 2*:**	Use the sum of angles identity to find $\tan(A + B)$. $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$ $= \frac{\frac{7}{24} + \frac{15}{8}}{1 - \frac{7}{24} \cdot \frac{15}{8}} = \frac{\frac{52}{24}}{1 - \frac{105}{192}} = \frac{416}{87}$

Exercise 3:	Find $\sin(A+B)$ given the following: $\sin A = \frac{4}{5}$ $90° < A < 180°$ $\cos B = \frac{3}{5}$ $0° < B < 90°$

***Rotational Transformation*:**	A point $P(x, y)$ can be rotated counterclockwise through an angle θ into $P'(x', x')$ using the following transformation: $$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
***Example 4*:**	Find the coordinates of the point $P(0, 4)$ after rotating it counterclockwise by 30°. $$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos 30 & -\sin 30 \\ \sin 30 & \cos 30 \end{bmatrix} \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$ $$= \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 2\sqrt{3} \end{bmatrix}$$ In the graph below, the red dot is the original point, and the blue dot is the transformed point. 

***Exercise 4*:**	Find the coordinates of the following points after rotating them through an angle of 60°: $A(0,4)$, $B(-\sqrt{3},1)$, $C(\sqrt{3},1)$, $O(0,2)$ Graph the four original points and the four transformed points. Use different colors for the original and transformed points. (Use a sheet of graph paper and attach it to these notes.) Using the point *O* as the center use a compass to draw a circle through the points *A*, *B*, and *C*. Then do the same thing with the transformed points.
Class work:	p 781: 1-13
Homework:	p 781: 14-38