Transition to College Math

Name ______________________________

Period ________

Date:

Unit **3: Trigonometry**

Lesson **4: Inverse Trigonometric Functions**

Essential Question: **Explain why inverse trigonometric relations are not necessarily functions. Give an example that is not in these notes.**

Standard: F-TF.6

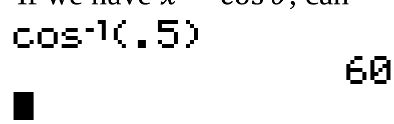
Understand that restricting a trigonometric function to a domain on which it is always increasing or always decreasing allows its inverse to be constructed.

Learning Target:

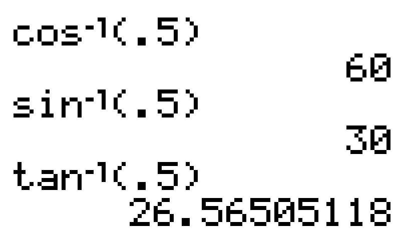
Evaluate inverse trigonometric functions. Use trigonometric equations and inverse trigonometric functions to solve problems. 80% of the students will be able to find θ if $\tan\theta = 1$.

Inverses of Trigonometric Functions:

We have seen that the output of the winding function is an ordered pair (x, y), where $x = \cos\theta$ and $y = \sin\theta$ and. The angle θ is the input to the winding function measured in radians. What about the inverse? If we have $x = \cos\theta$, can we find θ? Many calculators can calculate the inverse of the three trigonometric functions. For example, here is a screen shot of an HP-84 Plus calculation of the inverse of the cosine function.



We know that $\cos 60° = 0.5$. this shows that $\cos^{-1} 0.5 = 60°$. Similarly, $\sin^{-1} 0.5 = 30°$ and $\tan^{-1} 0.5 \cong 26.6°$ as shown in the second screen capture.



Summary

Not all inverses of functions are functions. Recall that a relation is not a function if an element of the domain maps onto two or more elements of the range. The third screen capture shows that the cosine of three different angles are all equal to the same value, 0.5. therefore, the inverse cosine is not a function. An angle of 60° equals $\frac{\pi}{3}$ $radians$. Based on the unit circle, angles of $\frac{\pi}{3}$, $-\frac{\pi}{3}$, and all angles that are coterminal with them have the same cosine, namely 0.5.

```
cos(60)
                    .5
cos(-60)
                    .5
cos(60+360)
                    .5
```

Using the notation, $\sin^{-1}(y)$, to represent the inverse of the sine function is convenient, but it could be confused with the reciprocal of the sine function. The reciprocal of the sine function is the cosecant function.

$$\sin^{-1}(y) \neq \csc(y)$$

***arcsine*:**

To avoid this confusion, we sometimes place the prefix "arc" in front of the name of the trigonometric function to indicate the inverse of the function:

arcsin(x)	pronounced, "arc sine of x".
arccos(x)	pronounced, "arc cosine of x".
arctan(x)	pronounced, "arc tangent of x".

Some calculators shorten the prefix "arc" to "a".

The prefix "arc" alludes to the fact that the angle measured in radians is the arc length on the unit circle. In this course, we will use the notation $\sin^{-1} y$ to denote the angle whose sine equals y.

Define the following terms.	
Function	
Trigonometric Functions	
Inverse	
arc sine	
arc cosine	
arc tangent	

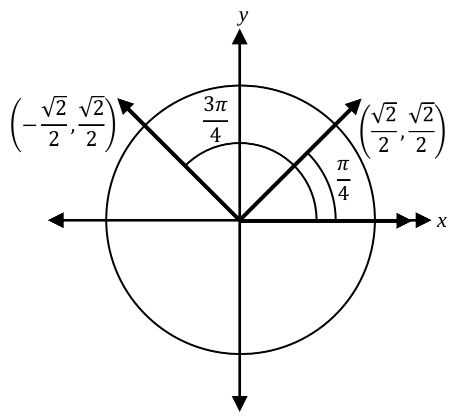
Example 1:

Find all possible angles whose sine equals $\frac{\sqrt{2}}{2}$. Specifically, find the set,

$$\left\{\theta \middle| \theta = \sin^{-1}\frac{\sqrt{2}}{2}\right\}$$

a. Find the values of $\theta: 0 \le \theta < 2\pi$ radians such that

$$\sin\theta = \frac{\sqrt{2}}{2}$$



From the table in the previous lesson, we see that,

$$\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

and

$$\sin\frac{3\pi}{4} = \frac{\sqrt{2}}{2}$$

b. Find the angles that are coterminal with angles measuring

$$\frac{\pi}{4} \quad \text{and} \quad \frac{3\pi}{4}$$

These are

$$\frac{\pi}{4} + (2\pi)n \quad \text{and} \quad \frac{3\pi}{4} + (2\pi)n$$

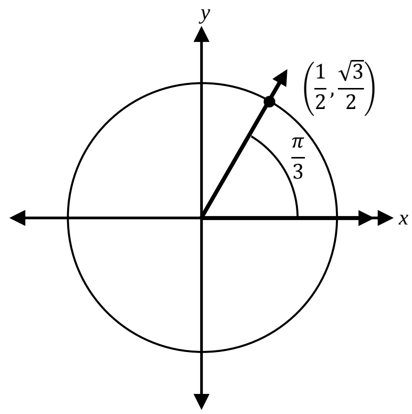
for all integral values of *n*.

***Exercise 1*:**	Find all possible values of $\theta = \tan^{-1}(1)$.
***Restricted Domains*:**	Having an infinite number of angles that satisfy the inverses of the trigonometric functions can cause problems. In fact these inverse relations are ***not functions***. In order to ensure that the inverse trigonometric relations are functions, we often restrict the domains of the trigonometric functions. To indicate the trigonometric functions with limited domains, we capitalize the name of the function. These functions and their domains are listed below: $\operatorname{Sin}\theta = \sin\theta \quad \left\{\theta \middle\vert -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right\}$ θ is restricted to Quadrants I and IV. $\operatorname{Cos}\theta = \cos\theta \quad \{\theta \vert 0 \le \theta \le \pi\}$ θ is restricted to Quadrants I and II. $\operatorname{Tan}\theta = \tan\theta \quad \left\{\theta \middle\vert -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right\}$ θ is restricted to Quadrants I and IV.

Inverse Trigonometric Functions:	While $\tan^{-1}(1)$ has many values, $\text{Tan}^{-1}(1)$ has only one, namely, $\frac{\pi}{4}$. We use these restricted domains to define ***inverse functions*** of the trigonometric functions. The inverse sine function is $\text{Sin}^{-1} a = \theta$ where $\text{Sin}\,\theta = a$ The inverse cosine function is $\text{Cos}^{-1} a = \theta$ where $\text{Cos}\,\theta = a$ The inverse tangent function is $\text{Tan}^{-1} a = \theta$ where $\text{Tan}\,\theta = a$

Symbol	Domain	Range
$\text{Sin}^{-1} a$	$\{a \mid -1 \leq a \leq 1\}$	$\left\{\theta \mid -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\right\}$
$\text{Cos}^{-1} a$	$\{a \mid -1 \leq a \leq 1\}$	$\{\theta \mid 0 \leq \theta \leq \pi\}$
$\text{Tan}^{-1} a$	$\{a \mid -\infty < a < \infty\}$	$\left\{\theta \mid -\frac{\pi}{2} < \theta < \frac{\pi}{2}\right\}$

Define the following terms.	
Domain	
Range	
Restricted Domain	
Compare and contrast the following pairs of terms	
Sin(θ) ***sin(θ)***	
Cos(θ) ***cos(θ)***	
Tan(θ) ***tan(θ)***	

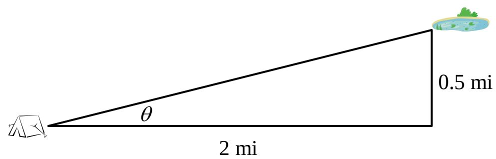
Example 2:	Evaluate each inverse trigonometric function. Use the unit circle table in the previous lesson. Give your answer in radians. a. $-\text{Cos}^{-1}\frac{1}{2}$ From the table in the previous lesson, we see that $\cos\dfrac{\pi}{3} = \dfrac{1}{2}$ and $\cos\dfrac{5\pi}{3} = \dfrac{1}{2}$. Since the range of $\text{Cos}^{-1}\, a$ is limited to $\{\theta \| 0 \leq \theta \leq \pi\}$, $\text{Cos}^{-1}\dfrac{1}{2} = \dfrac{\pi}{3}$.  b. $\text{Sin}^{-1}\, 2$ Since the domain of $\text{Sin}^{-1}\, a$ is $\{a \| -1 \leq a \leq 1\}$, $\text{Sin}^{-1}\, 2$ is undefined.

***Exercise 2*:**	Evaluate each inverse trigonometric function. Use the unit circle table in the previous lesson. Give your answer in radians. a. $\text{Sin}^{-1} -\frac{\sqrt{2}}{2}$. b. $\text{Cos}^{-1} 0$

Example 3:

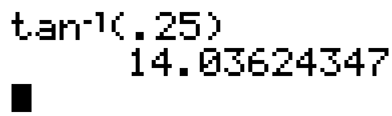
A group of hikers plans to walk from a campground to a lake. According to a map, the lake is 2 miles east and 0.5 miles north of the campground. If they want to take the most direct path, in what direction should the hikers head? Give your answer to the nearest degree.

a. Draw a figure that depicts all the information given.



b. Since the information given is length of the opposite side of the right triangle, and the length of the adjacent side of the right triangle, we should use the inverse tangent function.

$$\tan\theta = \frac{0.5}{2} = 0.25$$



$$\theta = \mathrm{Tan}^{-1}\, 0.25$$

$$\theta \cong 14°$$

The hikers should head 14° north of east.

Exercise 3:

Use the information given in Example 3 to answer the following:

An unusual rock formation is 1 mile east and 0.75 miles north of the lake. To the nearest degree, in what direction should the hikers head from the lake in order to reach the rock formation?

Example 4:

Solve each equation. Use the given restrictions on θ. Give your answer to the nearest degree.

a. $\cos\theta = 0.6 \qquad 0° \leq \theta \leq 180°$

The restrictions on θ are the same as those for the inverse cosine function. Using a calculator we find,

```
cos⁻¹(0.6)
         53.13010235
```

$\theta = \text{Cos}^{-1}(0.6) \cong 53.1°$

b. $\cos\theta = 0.6 \qquad 270° \leq \theta \leq 360°$

The restrictions on θ require that the terminal side of the angle be in Quadrant IV. Moreover, the reference angle is,

$$\theta_{ref} = \text{Cos}^{-1}(0.6) \cong 53.1°$$

Therefore,

$$\theta = 360° - \theta_{ref} \cong 360° - 53.1° \cong 306.9°$$

***Exercise 4*:**	Solve each equation to the nearest tenth. Use the given restrictions. a. $\tan\theta = -2$ $\quad -90° \leq \theta \leq 90°$ b. $\tan\theta = -2$ $\quad 90° \leq \theta \leq 180°$
Class work:	Inverse Trigonometric Functions Guided Practice Handout
Homework:	Inverse Trigonometric Functions Homework Handout