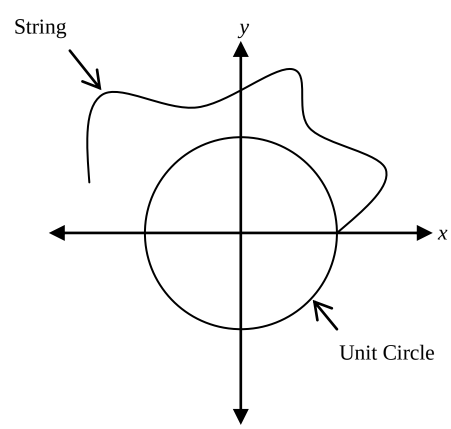
Transition to College Math

Name ________________________________

Period __________

Date: Unit **3: Trigonometry** Lesson **3: The Unit Circle**	Essential Question: **The Earth's orbit around the sun is nearly circular with a diameter of about 300 million kilometers. Assuming the Earth takes 365 days to complete one orbit, how long does the Earth take to travel 100 million kilometers in its orbit around the sun? Give your answer to the nearest day, and show your work.**
Standard: F-TF.1	Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.
***Learning Target*:**	Convert angle measures between radians and degrees. Find the values of trigonometric functions on the unit circle. 80% of the students will be able to convert 265° to radians.
***Unit Circle*:**	Consider the figure below: 
Summary	

	This figure depicts the coordinate axes with a circle centered on the origin. The radius of this circle is 1. Therefore, the circle is called the ***unit circle***. The figure also depicts a string attached to the unit circle where it intersects the positive *x*-axis. The coordinates where the string attaches are $(1, 0)$.
***Winding Function*:**	Now, suppose we stretch the string and wrap it counterclockwise around the unit circle. This defines the ***winding function***. The input to the winding function is the length of the string. The output of the winding function is the coordinates of the point where the end of the string lies, (x, y). We can define an angle of rotation whose terminal side contains the output point of the winding function. Moreover, the angle of rotation would pass the positive *x*-axis as many times as the string (winding function input) does. From the previous lesson, we know that the coordinates of the endpoint (*i.e.* the output of the winding function) is, $x = \cos\theta$ $y = \sin\theta$ θ is the angle of rotation.
***Radians*:**	In this example, we see that we can represent the angle in two ways: (1) the angle of rotation, and (2) the length of the string. We are all familiar with measuring angles using degrees, and we have extended it to angles greater than 360° and less than 0°. Using the length of the string input to the winding function is a fundamentally different concept: The angle is thought of as being the ***length***. We call this unit the ***radian***, and we set it equal to the length of the string that is the input to the winding function.

	From geometry, we know that the circumference of a circle with radius 1 is 2π. Therefore, we can readily determine the relationship between radians and degrees: $2\pi\ radians = 360\ degrees$ $radians = degrees \times \frac{\pi}{180}$ $degrees = radians \times \frac{180}{\pi}$ The following table summarizes the rules for converting between degrees and radians: **Degrees to Radians**: Multiply by $\frac{\pi\ radians}{180°}$ — **Radians to Degrees**: Multiply by $\frac{180°}{\pi\ radians}$ The concept of dividing a circle into 360 parts dates back to the ancient Babylonians, and we still use degrees to express the measure of an angle. Nevertheless as we shall see, radians are a very useful measure in science and engineering.
Define the following terms.	
unit circle	
winding function	
radian	

***Example 1*:**	a. Convert from degrees to radians. $45°$ $45° \times \frac{\pi\ radians}{180°} = \frac{\pi}{4} radians$ b. Convert from radians to degrees $-\frac{5\pi}{4} radians$ $-\frac{5\pi}{4} radians \times \frac{180°}{\pi\ radians} = -225°$
***Exercise 1*:**	Convert each angle measure from degrees to radians or from radians to degrees. a. $80°$ b. $\frac{\pi}{9} radians$ c. $-36°$ d. $4\pi\ radians$

Unit Circle:

By definition, the radius of the unit circle is one unit. Therefore, the distance from any point $P(x, y)$ on the unit circle to the origin is $r = 1$. Furthermore, any angle in standard position whose terminal side passes through point P:

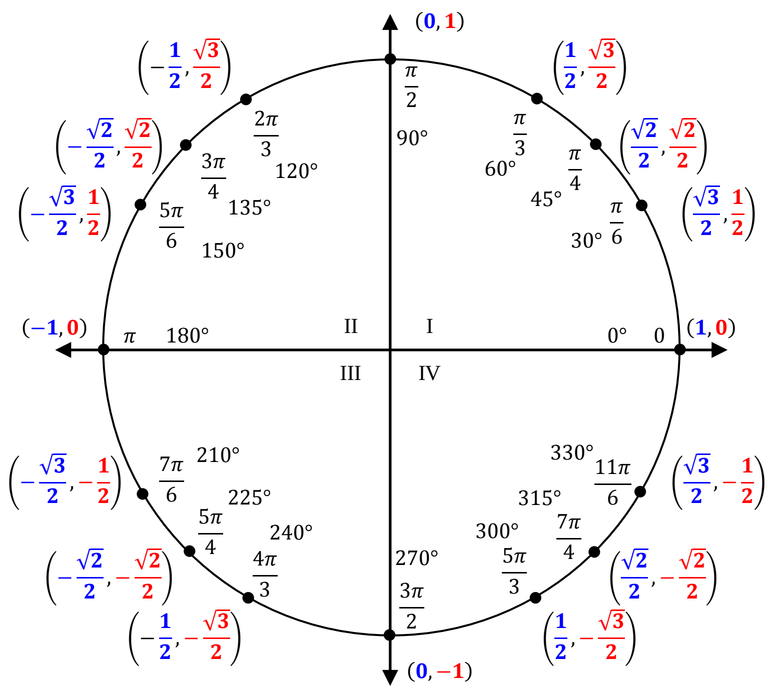
$$\sin\theta = \frac{y}{r} = \frac{y}{1} = y$$

$$\cos\theta = \frac{x}{r} = \frac{x}{1} = x$$

$$\tan\theta = \frac{y}{x}$$

The coordinates of P can be written as, $(\cos\theta, \sin\theta)$.

The diagram below depicts a unit circle labeled with the positions, coordinates, and angles of some commonly encountered angles.



The following table summarizes the information depicted on the unit circle diagram depicted above:

Degrees	**Radians**	**Coordinates**	**Degrees**	**Radians**	**Coordinates**
0	0	$(1,0)$	180	π	$(-1,0)$
30	$\frac{\pi}{6}$	$\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$	210	$\frac{7\pi}{6}$	$\left(-\frac{\sqrt{3}}{2},-\frac{1}{2}\right)$
45	$\frac{\pi}{4}$	$\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$	225	$\frac{5\pi}{4}$	$\left(-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)$
60	$\frac{\pi}{3}$	$\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$	240	$\frac{4\pi}{3}$	$\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$
90	$\frac{\pi}{2}$	$(0,1)$	270	$\frac{3\pi}{2}$	$(0,-1)$
120	$\frac{2\pi}{3}$	$\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)$	300	$\frac{5\pi}{3}$	$\left(\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$
135	$\frac{3\pi}{4}$	$\left(-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$	315	$\frac{7\pi}{4}$	$\left(\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)$
150	$\frac{5\pi}{6}$	$\left(-\frac{\sqrt{3}}{2},\frac{1}{2}\right)$	330	$\frac{11\pi}{6}$	$\left(\frac{\sqrt{3}}{2},-\frac{1}{2}\right)$
180	π	$(-1,0)$	360	2π	$(1,0)$

***Example 2*:**	Use the unit circle table above to find the exact value of each trigonometric function. a. $\cos 210°$ From the table, the coordinates of the point on the unit circle intercepted by an angle of 210° is $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$. $\cos 210° = x = -\frac{\sqrt{3}}{2}$ b. $\tan \frac{5\pi}{3}$ From the table, the coordinates of the point on the unit circle intercepted by an angle of $\frac{5\pi}{3} radisns$ is $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$. $\tan \frac{5\pi}{3} = \frac{y}{x} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\frac{\sqrt{3}}{2} \cdot \frac{2}{1} = -\sqrt{3}$
***Exercise 2*:**	Use the unit circle table above to find the exact value of each trigonometric function. a. $\sin 315°$ b. $\tan 180°$ c. $\cos \frac{4\pi}{3}$

***Reference Angles*:**

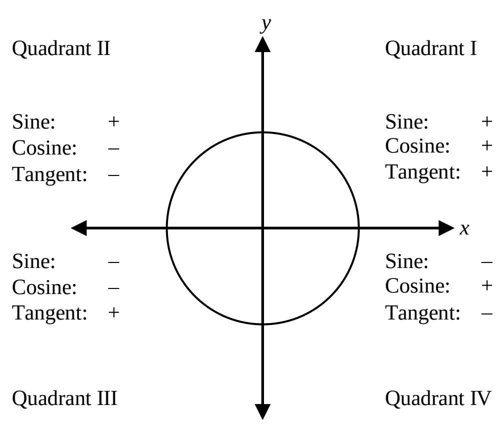
Recall the definition of reference angles, "For an angle in standard position, the reference angle is the positive, acute angle formed by the terminal side and the *x*-axis." You can use the reference angle and the table above to find the trig functions of any angle. The reference angle always satisfies the following inequality:

$$0 \leq \theta \leq \frac{\pi}{2}$$

To find the sine, cosine, or tangent of any angle θ, do the following:

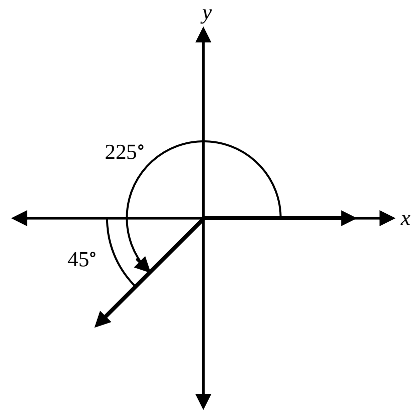
1. Determine the reference angle of the angle θ.
2. Use Quadrant I of the unit circle to find the sine, cosine, or tangent of the reference angle.
3. Determine the quadrant of the terminal side of the angle in standard position.
4. Change the sign of the sine, cosine, or tangent of the angle, if necessary, based on the quadrant of the terminal side.

Use the following figure to determine whether or not to change the sign of the sine, cosine, or tangent of the angle:



***Example 3*:**	Use a reference angle to find the exact value of the sine, cosine, and tangent of 225°.

1. Find the reference angle.



The reference angle measures 45°.

2. Find the sine, cosine, and tangent of the reference angle.

$$\sin 45° = \frac{\sqrt{2}}{2}$$

$$\cos 45° = \frac{\sqrt{2}}{2}$$

$$\tan 45° = 1$$

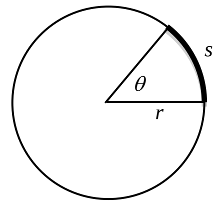
3. Change the signs as needed. In Quadrant III, the sine and cosine are negative, and the tangent is positive.

$$\sin 225° = \frac{-\sqrt{2}}{2}$$

$$\cos 225° = -\frac{\sqrt{2}}{2}$$

$$\tan 225° = 1$$

***Exercise 3*:**	Use a reference angle to find the exact values of the sine, cosine, and tangent of each angle. a. $270°$ b. $\frac{11\pi}{6}$ c. $-30°$

***Arc Length*:**	A ***central angle*** is defined as follows: "The angle subtended at the center of a circle by two given points on the circle."[1] If you know the central angle, the arc length intercepted by the central angle is defined by the following relation:  $$\frac{radian\ measure\ of\ \theta}{radian\ measure\ of\ circle} = \frac{arc\ length\ intercepted\ by\ \theta}{circumfrence\ of\ circle}$$ This can be written symbolically, $$\frac{\theta}{2\pi} = \frac{s}{2\pi r}$$ **Arc Length Formula** For a circle of radius r, the arc length s intercepted by a central angle θ (measured in radians) is given by, $$s = r\theta$$

[1] http://www.mathopenref.com/circlecentral.html

***Example 4*:**	A human centrifuge is a device used in training astronauts. It simulates the intense g-forces experienced during a rocket launch. The passenger cab of a particular centrifuge traces out a circle with a diameter of 58 ft, and it makes 32 complete revolutions about the central hub in 1 minute. To the nearest foot, how far does an astronaut travel in 1 second? 1. Find the radius of the centrifuge. $r = \frac{58}{2} = 29\ ft$ 2. Find the angle through which the cab rotates in 1 second. $\frac{radians\ rotated\ in\ 1\ second}{1 second} = \frac{radians\ rotated\ in\ 60\ seconds}{60\ seconds}$ $\frac{\theta\ radians}{1\ s} = \frac{32(2\pi)\ radians}{60\ s}$ $\theta = \frac{32 \cdot 2\pi}{60\ s} \cdot 1\ s$ $\theta = \frac{16\pi}{15}$ 3. Find the length of the arc intercepted by $16\pi/15$ radians. $s = r\theta$ $s = 29\left(\frac{16\pi}{15}\right)$ $s \cong 97$ In 1 second, the astronaut travels approximately 97 feet.

Exercise 4:	An hour hand on Big Ben's Clock Tower in London is 14 ft long. To the nearest tenth of a foot, how far does the tip of the hour hand travel in 1 minute?
Class work:	Unit Circle Guided Practice Handout
Homework:	Unit Circle Homework Handout