Transition to College Math

Name ___________________________________

Period __________

Date: Unit **2: Matrices** Lesson **4: Inverse Matrices**	Essential Question: **Why must a matrix have a non-zero determinant in order for it to have an inverse?**
Standard: N-VM.10	Understand that the zero and identity matrices play a role in matrix addition and multiplication similar to the role of 0 and 1 in the real numbers. The determinant of a square matrix is nonzero if and only if the matrix has a multiplicative inverse.
***Learning Target*:**	The student will learn how to find the inverse of a square matrix. 80% of the students will be able to find the inverse of the following matrix: $\begin{bmatrix} -3 & -5 \\ 2 & 3 \end{bmatrix}$ In addition to donuts, Jane and Jack's Donut Shoppe sells muffins and bagels. One a certain day, Guinevere bought two bagels and three muffins for \$8.00. Andromeda bought, three bagels three muffins, and two donuts for \$10.00. Robert bought one muffin, five bagels, and four donuts for \$13.00. What is the price of donuts, muffins, and bagels? You recognize this problem as a system of equations, $2m + 3b + 0d = 8$ $3m + 3b + 2d = 10$ $1m + 5b + 4d = 13,$ Where *m* is the price of a muffin, *b* is the price of a bagel, and *d* is the price of a donut.
Summary	

	Rather than using substitution or elimination to solve this problem, we will use matrix algebra. The first step is to cast this system of equations as a matrix equation. We create a 3×3 matrix using the coefficients of the system of equations. $\begin{bmatrix} 2 & 3 & 0 \\ 3 & 3 & 2 \\ 1 & 5 & 4 \end{bmatrix}$ Then we multiply a column matrix consisting of the unknown prices and set it equal to the amount each person spent. $\begin{bmatrix} 2 & 3 & 0 \\ 3 & 3 & 2 \\ 1 & 5 & 4 \end{bmatrix} \begin{bmatrix} m \\ b \\ d \end{bmatrix} = \begin{bmatrix} 8 \\ 10 \\ 13 \end{bmatrix}$ Now, we simply solve this equation for the column matrix $\begin{bmatrix} m \\ b \\ d \end{bmatrix}$. In this lesson, we will learn how to solve this type of matrix equation.
***Inverse*:**	Most real numbers have an ***inverse*** under multiplication. That is, for a real number a, $a \cdot a^{-1} = 1$. There are two things in this equation to note: The symbol a^{-1} represents the inverse of a, and $a^{-1} = 1/a$. The symbol 1 is the identity under multiplication. ***Most*** real numbers have a multiplicative inverse, but not all. The number 0 has no multiplicative inverse. That is, there is no real number a such that, $0 \cdot a = 1$.
***Inverse Matrices*:**	In the same way many square matrices have a multiplicative inverse: $\mathbf{A} \cdot \mathbf{A}^{-1} = \mathbf{A}^{-1} \cdot \mathbf{A} = \mathbf{I}$ The matrix **I** is the identity matrix, and it is a square matrix with all the diagonal terms equal to 1 and all the off-diagonal terms equal to 0. An identity matrix may have any number of rows, but it is always square.

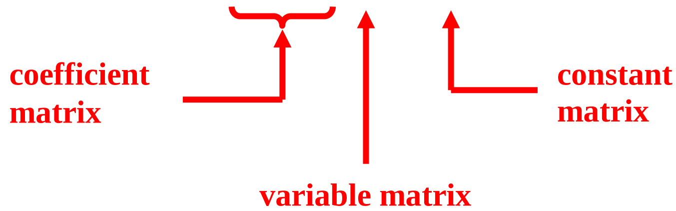
***Identity matrix*:**	
2×2	$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$
3×3	$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$
4×4	$\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$
***Inverse Matrix*:**	Consider this matrix product, $\begin{bmatrix}-5 & 2\\-8 & 3\end{bmatrix}\cdot\begin{bmatrix}3 & -2\\8 & -5\end{bmatrix}=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ From this matrix equation, it is clear that $\begin{bmatrix}-5 & 2\\-8 & 3\end{bmatrix}^{-1}=\begin{bmatrix}3 & -2\\8 & -5\end{bmatrix}$ and $\begin{bmatrix}3 & -2\\8 & -5\end{bmatrix}^{-1}=\begin{bmatrix}-5 & 2\\-8 & 3\end{bmatrix}$
***Finding the Inverse of a 2×2 Matrix*:**	The relationship between the two matrices above is apparent. The diagonal terms are interchanged, and the off diagonal terms have their signs changed. However, there is one more step. You must divide by the determinant. For both of these matrices, the determinant is 1, and dividing by 1 makes no changes. In general, the inverse of a 2×2 matrix is, $\begin{bmatrix}a & b\\c & d\end{bmatrix}^{-1}=\frac{1}{ad-bc}\begin{bmatrix}d & -b\\-c & a\end{bmatrix}$

Solving Systems of Equations:	Suppose we have the following system of equations: $2x - 3y = 4$ $3x - 4y = 7$ If we solve the first equation for *x*, we can substitute the expression for *x* into the second equation. Then we can solve for *y*. Finally we substitute the value for *y* into the equation for *x*. $2x = 3y + 4$ $x = \frac{3}{2}y + 2$ $3\left(\frac{3}{2}y + 2\right) - 4y = 7$ $\frac{9}{2}y + 6 - 4y = 7$ $\frac{9}{2}y - \frac{8}{2}y = 7 - 6 = 1$ $\frac{1}{2}y = 1$ $y = 2$ $x = \frac{3}{2}(2) + 2 = 5$ The solution of this system of equations is $(5, 2)$. We can check the answer by substituting the values into the original equations. $2 \cdot 5 - 3 \cdot 2 = 10 - 6 = 4$ ✓ $3 \cdot 5 - 4 \cdot 2 = 15 - 8 = 7$ ✓

***Matrix Equation*:**

This system of equations can be written as a single matrix equation.

$$\begin{bmatrix} 2 & -3 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$



We can solve this equation by multiplying this matrix equation on the left side by the inverse of the coefficient matrix.

$$\begin{bmatrix} 2 & -3 \\ 3 & -4 \end{bmatrix}^{-1} \begin{bmatrix} 2 & -3 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 3 & -4 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 3 & -4 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 3 & -4 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

Now, we find the inverse of the coefficient matrix.

$$\begin{bmatrix} 2 & -3 \\ 3 & -4 \end{bmatrix}^{-1} = \frac{1}{2(-4) - 3(-3)} \begin{bmatrix} -4 & 3 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} -4 & 3 \\ -3 & 2 \end{bmatrix}$$

Finally,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4 & 3 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \end{bmatrix} = \begin{bmatrix} -4 \cdot 4 + 3 \cdot 7 \\ -3 \cdot 4 + 2 \cdot 7 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

You can solve a system of equations of any size using a matrix equation. The only requirement is that the inverse of the coefficient matrix exist and that you can find it.

Define the following terms:	
Solve Equation	
Coefficient Matrix	
Variable Matrix	
Constant Matrix	
Matrix Equation	

***Systems of Equations*:**

We have learned how to solve such systems of equations by using substitution or elimination methods, but matrices provide us with another, more powerful tool to solve systems of equations. We rewrite the system of equations as a matrix equation, and solve the matrix equation.

A matrix equation contains three matrices:

1. Square coefficient matrix
2. Column variable matrix
3. Column constant matrix

The equation itself is,

[coefficient matrix]×[variable matrix]=[constant matrix]

The coefficient matrix is a square matrix made from the coefficients of the variables in the original equations. The coefficient matrix for the example above would be,

$$\begin{bmatrix} 3 & 1 & -2 \\ 1 & 2 & 1 \\ 5 & -2 & -1 \end{bmatrix}$$

The variable matrix is a column matrix containing the variables of interest. The variable matrix for the example above would be,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

The constant matrix is a column matrix containing the constants from the system of equations. The constant matrix for the example above would be,

$$\begin{bmatrix} -3 \\ -1 \\ 7 \end{bmatrix}$$

	Therefore, the matrix equation for the example above would be, $$\begin{bmatrix} 3 & 1 & -2 \\ 1 & 2 & 1 \\ 5 & -2 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \\ 7 \end{bmatrix}$$
In your own words, define the following terms:	
Polynomial	
Coefficient	
Root	
Complex Number	

***Example 1*:**

Write the matrix equation for the following system of linear equations:

$$3a + 2b - 5c + d = -6$$

$$-a + 3b - c + 4d = 11$$

$$2a - 2b - c + 3d = 19$$

$$-3a + 2b - c - d = -14$$

Each of the four equations has four, constant coefficients. We write the coefficients of the first equation as the elements in the first row of the coefficient matrix. Then we write the coefficients of the second equation in the second row…

$$\begin{bmatrix} 3 & 2 & -5 & 1 \\ -1 & 3 & -1 & 4 \\ 2 & -2 & -1 & 3 \\ -3 & 2 & -1 & -1 \end{bmatrix}$$

The system has four variables. We write these as a column matrix. Notice that the first variable must be the top entry in the variable matrix, etc.

We can rewrite the system of equations as the following matrix equation:

$$\begin{bmatrix} 3 & 2 & -5 & 1 \\ -1 & 3 & -1 & 4 \\ 2 & -2 & -1 & 3 \\ -3 & 2 & -1 & -1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -6 \\ 11 \\ 19 \\ -14 \end{bmatrix}$$

Suppose we multiple both sides of this matrix equation on the left by the ***inverse matrix*** of the coefficient matrix.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 3 & 2 & -5 & 1 \\ -1 & 3 & -1 & 4 \\ 2 & -2 & -1 & 3 \\ -3 & 2 & -1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} -6 \\ 11 \\ 19 \\ -14 \end{bmatrix}$$

The matrix on the left is the ***identity matrix***. It is analogous to the real number 1.

The problem reduces to finding the inverse matrix. Before we learn how to find an inverse matrix, let's use a graphing calculator to solve this matrix equation.

The easiest way to solve this matrix equation is to use technology. I used a graphing calculator.

First, I entered the coefficients into a 4×4 matrix:

[A]

$$\begin{bmatrix} 3 & 2 & -5 & 1 \\ -1 & 3 & -1 & 4 \\ 2 & -2 & -1 & 3 \\ -3 & 2 & -1 & -1 \end{bmatrix}$$

Then I entered the constants into a 4×1 matrix:

[C]

$$\begin{bmatrix} -6 \\ 11 \\ 19 \\ -14 \end{bmatrix}$$

Then I multiplied the inverse of the coefficient matrix and the constant matrix:

$[A]^{-1}[C]$

$$\begin{bmatrix} 1 \\ -2 \\ 2 \\ 5 \end{bmatrix}$$

It is easy to check the answer. For example, the first equation becomes:

$$3(1) + 2(-2) - 5(2) + (5) = -6$$

$$3 - 4 - 10 + 5 = -6 \quad ✓$$

***Exercise 1*:**	a. Use a matrix equation to solve the following system of equations: $2x + y = 2$ $3x + 2y = 2$ b. Some Foods sells cashews and pecans by the pound. Jennifer bought two pounds of cashews and two pounds of pecans for $23.50. Samuel bought three pounds of cashews and one pound of pecans for $22.25. Set up the matrix equation and solve it to find the unit price for cashews and pecans.

***Singular Matrix*:**	Consider the following matrix: $\begin{bmatrix} 3 & -6 \\ 4 & -8 \end{bmatrix}$ Find its inverse. $\begin{bmatrix} 3 & -6 \\ 4 & -8 \end{bmatrix}^{-1} = \frac{1}{\begin{vmatrix} 3 & -6 \\ 4 & -8 \end{vmatrix}} \begin{bmatrix} -8 & 6 \\ -4 & 3 \end{bmatrix} = \frac{1}{-24+24} \begin{bmatrix} -8 & 6 \\ -4 & 3 \end{bmatrix}$ $\begin{bmatrix} 3 & -6 \\ 4 & -8 \end{bmatrix}^{-1} = \frac{\mathbf{1}}{\mathbf{0}} \begin{bmatrix} -8 & 6 \\ -4 & 3 \end{bmatrix}$ Since we cannot divide by 0, this matrix has no inverse. We say that it is ***singular***.
***Inverse Matrix*:**	For every square matrix **A** there is another square matrix $\mathbf{A^{-1}}$ that has the same dimensions, and $\mathbf{A \cdot A^{-1} = A^{-1} \cdot A = I}$ If and only if **det(A)** $\neq$ **0**.
***Example 2*:**	Find the inverse of the following matrix, if it exists. $\begin{bmatrix} 9 & 6 \\ 24 & 16 \end{bmatrix}$ First, find the determinant: $\begin{vmatrix} 9 & 6 \\ 24 & 16 \end{vmatrix} = 9 \cdot 16 - 24 \cdot 6 = 144 - 144 = 0$ Since the determinant equals 0, this matrix is singular and has no inverse.

Exercise 2:	Find the inverse of each of the following matrices, if it exists. a. $\begin{bmatrix} -3 & 2 \\ 3 & 3 \end{bmatrix}$ b. $\begin{bmatrix} -10 & 2 \\ 5 & -1 \end{bmatrix}$ c. $\begin{bmatrix} -10 & -4 \\ -9 & 3 \end{bmatrix}$
Class work:	Guided Practice handout
Homework:	Homework handout