Transition to College Math

Name ________________________________

Period __________

Date: Unit **2: Matrices and Vectors** Lesson **3: Linear Transformations and Determinants**	Essential Question: **Does the diagonal rule correctly calculate the determinant of a 2×2 matrix? Explain.**
Standard: N-VM.12	Work with 2 × 2 matrices as transformations of the plane, and interpret the absolute value of the determinant in terms of area.
***Learning Target*:**	Learn to calculate the determinant of a square matrix of arbitrary dimensions. Recognize that a parallelogram is the result of a linear transformation of a unit square. Moreover, the area of the parallelogram is the absolute value of the determinant of the transformation matrix. 80% of the students will be able to find the determinant of the following matrix: $\begin{vmatrix} 1 & 6 & 7 \\ -2 & -5 & -8 \\ 4 & 4 & 9 \end{vmatrix}$
***Determinants*:**	Every square matrix has a ***determinant***. The determinant of a matrix is analogous to the absolute value of a real number. Nevertheless, there are some significant differences. While the absolute value of a real number can never be negative, the determinant of a matrix can be positive, negative, or zero.
***Order of a Determinant*:**	The determinant of a 2×2 matrix is a ***second-order-determinant***. Similarly, the determinant of a 3×3 matrix is a ***third-order-determinant***.
Summary	

Notation:	Suppose **A** is a square matrix, the determinant of **A** can be written in two ways, det **A** = **|A|**.
Second Order Determinant:	The second-order-determinant is the difference of the diagonal products, $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$
Example 1:	Evaluate the following determinants: a. $\begin{vmatrix} 8 & -7 \\ 0 & -2 \end{vmatrix}$ $\begin{vmatrix} 8 & -7 \\ 0 & -2 \end{vmatrix} = 8(-2) - (-7)0$ $= -16 - 0$ $= -16$ b. $\begin{vmatrix} 4 & -9 \\ -3 & 9 \end{vmatrix}$ $\begin{vmatrix} 4 & -9 \\ -3 & 9 \end{vmatrix} = 4 \cdot 9 - (-9)(-3)$ $= 36 - 27$ $= 9$ c. $\begin{vmatrix} -6 & 6 \\ 9 & -5 \end{vmatrix}$ $\begin{vmatrix} -6 & 6 \\ 9 & -5 \end{vmatrix} = (-6)(-5) - 6 \cdot 9$ $= 30 - 54$ $= -24$

Exercise 1:	Evaluate the following determinants: a. $\begin{vmatrix} -2 & 3 \\ 4 & -2 \end{vmatrix}$ b. $\begin{vmatrix} -7 & -9 \\ 4 & -9 \end{vmatrix}$ c. $\begin{vmatrix} -26 & -98 \\ 87 & -78 \end{vmatrix}$

Third Order Determinants:	The determinant of a 3×3 matrix is a ***third order determinant***. You can use the ***diagonal rule*** to find the determinant of a 3×3 matrix. The diagonal rule has the following steps: 1. Write the determinant and rewrite the first two columns to the right of the determinant. 2. Beginning in the upper left corner, draw three lines diagonally down and to the right. 3. Find the product of the three numbers on each diagonal line. 4. Beginning in the upper right corner, draw three lines diagonally down and to the left. 5. Find the product of the three numbers on each diagonal line. 6. Add the three products from step 3. 7. Add the three products from step 5. 8. Subtract the sum from step 7 from the sum from step 6. 9. The result from step 8 is the determinant of the 3×3 matrix. Example 2 on the next page illustrates this process.

Example 2:

Evaluate $\begin{vmatrix} -2 & -5 & 2 \\ 2 & 6 & 5 \\ -1 & 8 & 2 \end{vmatrix}$ using the diagonal rule.

1. Write the determinant and rewrite the first two columns to the right of the determinant.

$$\begin{vmatrix} -2 & -5 & 2 \\ 2 & 6 & 5 \\ -1 & 8 & 2 \end{vmatrix} \begin{matrix} -2 & -5 \\ 2 & 6 \\ -1 & 8 \end{matrix}$$

2. Beginning in the upper left corner, draw three lines diagonally down and to the right.

$$\begin{vmatrix} -2 & -5 & 2 \\ 2 & 6 & 5 \\ -1 & 8 & 2 \end{vmatrix} \begin{matrix} -2 & -5 \\ 2 & 6 \\ -1 & 8 \end{matrix}$$

3. Find the product of the three numbers on each diagonal line.

$(-2) \cdot 6 \cdot 2 = -24$

$(-5) \cdot 5 \cdot (-1) = 25$

$2 \cdot 2 \cdot 8 = 32$

4. Beginning in the upper right corner, draw three lines diagonally down and to the left.

$$\begin{vmatrix} -2 & -5 & 2 \\ 2 & 6 & 5 \\ -1 & 8 & 2 \end{vmatrix} \begin{matrix} -2 & -5 \\ 2 & 6 \\ -1 & 8 \end{matrix}$$

5. Find the product of the three numbers on each diagonal line.

$2 \cdot 6 \cdot (-1) = -12$

$(-2) \cdot 5 \cdot 8 = -80$

$(-5) \cdot 2 \cdot 2 = -20$

	6. Add the three products from step 3. $-24 + 25 + 32 = 33$ 7. Add the three products from step 5. $-12 + (-80) + (-20) = -112$ 8. Subtract the sum from step 7 from the sum from step 6. $33 - (-112) = 145$ 9. The result from step 8 is the determinant of the 3×3 matrix. $\begin{vmatrix} -2 & -5 & 2 \\ 2 & 6 & 5 \\ -1 & 8 & 2 \end{vmatrix} = 145$
***Exercise 2*:**	1. Evaluate $\begin{vmatrix} -3 & -1 & 6 \\ 7 & -5 & 4 \\ -1 & 0 & -5 \end{vmatrix}$ using the diagonal rule.

Exercise 2 continued:	2. Evaluate $\begin{vmatrix} -8 & -3 & 3 \\ -7 & 3 & -9 \\ 2 & -4 & -9 \end{vmatrix}$ using the diagonal rule.

***Cofactor*:**

The diagonal rule works for 3×3 matrices, but it ***does not work for any other size matrix***. The easiest way to find the determinant of a matrix larger than 3×3 is to use ***cofactors***.

Consider the following determinant and the circled element:

$$\begin{vmatrix} 8 & 6 & -5 & -8 \\ -5 & 4 & \textcircled{-8} & -5 \\ 2 & 6 & 4 & -2 \\ 8 & 0 & 3 & -3 \end{vmatrix}$$

The circled element, –8, is in the 2nd row and 3rd column. The cofactor is the determinant formed by eliminating the 2nd row and 3rd column and multiplying by $(-1)^{2+3}$.

$$(-1)^{2+3}\begin{vmatrix} 8 & 6 & -8 \\ 2 & 6 & -2 \\ 8 & 0 & -3 \end{vmatrix}.$$

In order to find the determinant, find all the cofactors of any single row or column in the determinant. Multiply the cofactors by their associated element. Add the resulting products.

***Example 3*:**

To illustrate this process, let's use cofactors to find the determinant of Example 2.

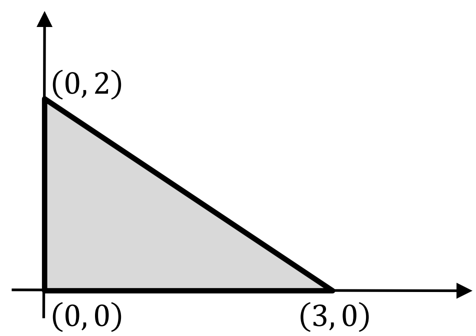
$$\begin{vmatrix} -2 & -5 & 2 \\ 2 & 6 & 5 \\ -1 & 8 & 2 \end{vmatrix}$$

$$= 2(-1)\begin{vmatrix} -5 & 2 \\ 8 & 2 \end{vmatrix} + 6(+1)\begin{vmatrix} -2 & 2 \\ -1 & 2 \end{vmatrix} + 5(-1)\begin{vmatrix} -2 & -5 \\ -1 & 8 \end{vmatrix}$$

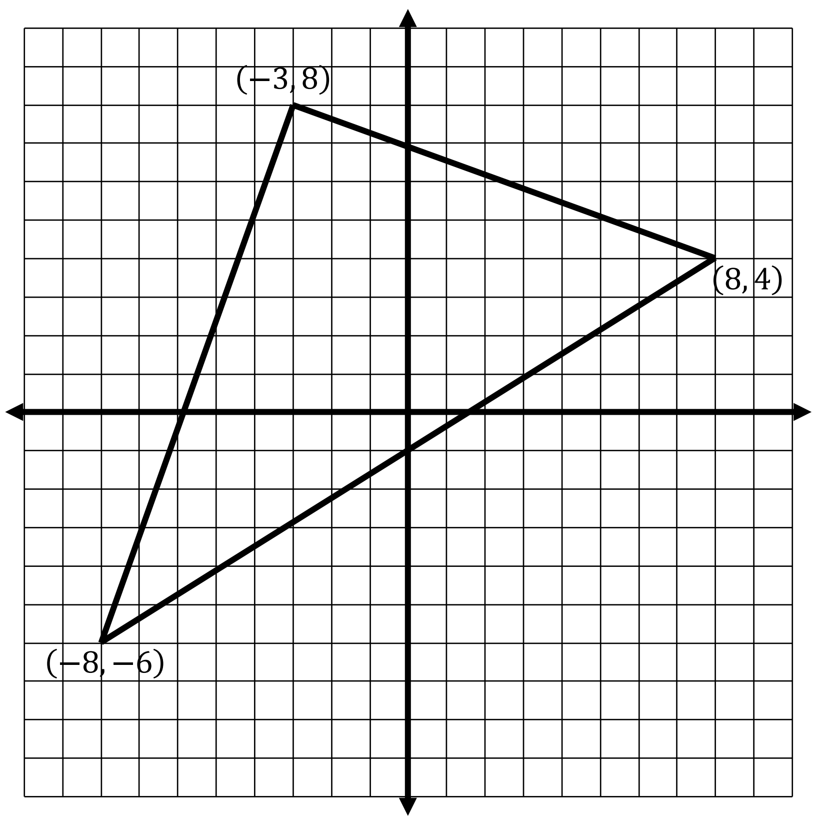
$$= -2(-5\cdot 2 - 2\cdot 8) + 6\big(-2\cdot 2 - 2(-1)\big) - 5\big(-2\cdot 8 - (-5)(-1)\big)$$

$$= -2(-26) + 6(-2) - 5(-21)$$

$$= 52 - 12 + 105 = 145$$

Area of a Triangle:	Determinants can be used to find the area of a triangle from the coordinates of the triangle's vertices. Suppose a triangle has the following coordinates: (a, b), (c, d), and (e, f). Then the area of the triangle is given by the following formula: $$A = \pm\frac{1}{2}\begin{vmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{vmatrix}$$ Depending on the order of the vertices in the determinant, the determinant could be positive or negative. The "±" indicates that the area should not be negative. Let's look at two examples.
***Example 3*:**	Find the area of the following triangle:  Using the formula above, $$A = \pm\frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ 0 & 2 & 1 \\ 3 & 0 & 1 \end{vmatrix}$$ If we use the cofactors of the first column to find the determinant, then the area is, $$A = \pm\frac{1}{2}\left(0 - 0 + 3\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix}\right)$$ $$A = \pm\frac{1}{2}\cdot 3(0\cdot 1 - 1\cdot 2) = \pm\frac{1}{2}(-6)$$

	Since the area must be positive, $$A = 3.$$ Of course, we know the area of a triangle is, $$A = \frac{1}{2}bh.$$ In this case, this formula is simple: $$A = \frac{1}{2}bh = \frac{1}{2} \cdot 2 \cdot 3 = 3.$$
***Example 3, continued*:**	However, the determinant can be used to find the area of a triangle that does not lend itself easily to the formula you learned in geometry. Consider the triangle depicted below:

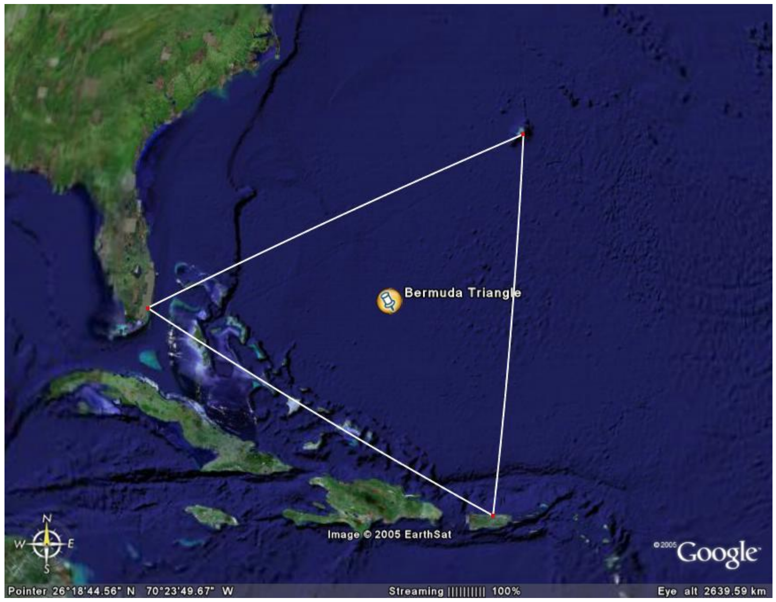


	The determinant that gives the area of this triangle is, $$A = \pm\frac{1}{2}\begin{vmatrix} -8 & -6 & 1 \\ 8 & 4 & 1 \\ -3 & 8 & 1 \end{vmatrix}$$ Let's find the determinant by using the cofactors of the third column. $$A = \pm\frac{1}{2}\left(\begin{vmatrix} 8 & 4 \\ -3 & 8 \end{vmatrix} - \begin{vmatrix} -8 & -6 \\ -3 & 8 \end{vmatrix} + \begin{vmatrix} -8 & -6 \\ 8 & 4 \end{vmatrix}\right)$$ $$A = \pm\frac{1}{2}\left[\left(8\cdot 8 - 4(-3)\right) - \left(-8\cdot 8 - (-6)(-3)\right) + \left(-8\cdot 4 - (-6)8\right)\right]$$ $$A = \pm\frac{1}{2}[(64+12) - (-64-18) + (-32+48)]$$ $$A = \pm\frac{1}{2}[76+82+16] = 87$$ The area of the triangle is $A = 87$.
***Exercise 3*:**	The Bermuda Triangle is a triangular area of the Atlantic ocean with vertices at Bermuda, Miami, FL, and San Juan, PR. Many mysterious occurrences are said to have happened inside the Bermuda Triangle. One of the better known is the disappearance of Flight 19. "At about 2:10 p.m. on the afternoon of 5 December 1945, Flight 19, consisting of five TBM Avenger Torpedo Bombers…departed from the U. S. Naval Air Station, Fort Lauderdale, Florida…" "A radio message intercepted at about 4 p.m. was the first indication that Flight 19 was lost. This message, believed to be between the leader on Flight 19 and another pilot in the same flight, indicated that the instructor was uncertain of his position and the direction of the Florida coast. The aircraft also were experiencing malfunction of their compasses. Attempts to establish communications on the training frequency were

	unsatisfactory due to interference from Cuba broadcasting stations, static, and atmospheric conditions. All radio contact was lost before the exact nature of the trouble or the location of the flight could be determined. Indications are that the flight became lost somewhere east of the Florida peninsula and was unable to determine a course to return to their base. The flight was never heard from again and no trace of the planes were ever found. It is assumed that they made forced landings at sea, in darkness somewhere east of the Florida peninsula, possibly after running out of gas. It is known that the fuel carried by the aircraft would have been completely exhausted by 8 p.m. The sea in that presumed area was rough and unfavorable for a water landing. It is also possible that some unexpected and unforeseen development of weather conditions may have intervened although there is no evidence of freak storms in the area at the time." "All available facilities in the immediate area were used in an effort to locate the missing aircraft and help them return to base. These efforts were not successful. No trace of the aircraft was ever found even though an extensive search operation was conducted until the evening of 10 December 1945, when weather conditions deteriorated to the point where further efforts became unduly hazardous. Sufficient aircraft and surface vessels were utilized to satisfactorily cover those areas in which survivors of Flight 19 could be presumed to be located." "One search aircraft was lost during the operation. A PBM patrol plane which was launched at approximately 7:30 p.m., 5 December 1945, to search for the missing TBM's. This aircraft was never seen nor heard from after take-off. Based upon a report from a merchant ship off Fort Lauderdale which sighted a "burst of flame, apparently an explosion, and passed through on oil slick at a time and place which matched the presumed location of the PBM, it is believed this aircraft exploded at sea and sank at approximately 28.59 N; 80.25 W. No trace of the plane or its crew was ever found."[1]

[1] http://www.history.navy.mil/faqs/faq15-1.htm

The illustration below depicts the location of the Bermuda Triangle.



The latitude and longitude of the vertices are[2],

Bermuda: (32.307800, -64.7505)

Miami: (25.761680, -80.19179)

San Juan: (18.466334, -66.105722)

In order to use the coordinates as a reference point, we must first convert them to degrees.

Pointer: (26.312378, -70.397131)

In order to find the area of the Bermuda Triangle, the easiest way is to first convert the coordinates of the vertices to coordinates relative to the pointer.

Bermuda: (,)

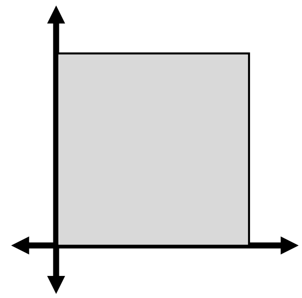
Miami: (,)

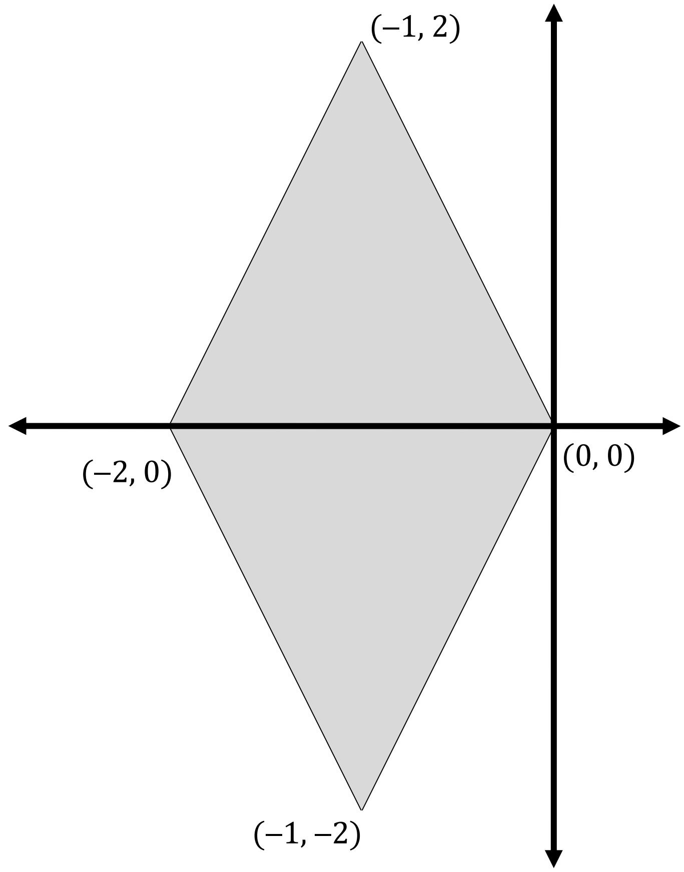
San Juan: (,)

[2] http://www.latlong.net/

	"A **nautical mile** (symbol **M**, **NM** or **nmi**) is a unit of distance that is approximately one minute of arc measured along any meridian."[3] Each degree of latitude is approximately 60 nautical miles. At the latitude of the pointer, each degree of longitude is approximately 53.78 nautical miles. Use these conversion factors to convert the coordinates from degrees to nautical miles. (The first coordinate is degrees of latitude, and the second coordinate is degrees of longitude.) Bermuda: (,) Miami: (,) San Juan: (,) Now, write the 3×3 determinate from these coordinates and calculate the area in nmi^2.

[3] http://en.wikipedia.org/wiki/Nautical_mile

Linear Transformations:	Consider the following unit square.  Since all four sides are one unit long, the vertices have the following coordinates: $(0, 0), (1, 0), (1,1)$, and $(0, 1)$ Let's represent these coordinates as a matrix. $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. Now, let's perform the following matrix multiplication, $\begin{bmatrix} -1 & -1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -2 & -1 \\ 0 & 2 & 0 & -2 \end{bmatrix}$. This last matrix represents the coordinates of a quadrilateral. Let's graph this quadrilateral on the next page.



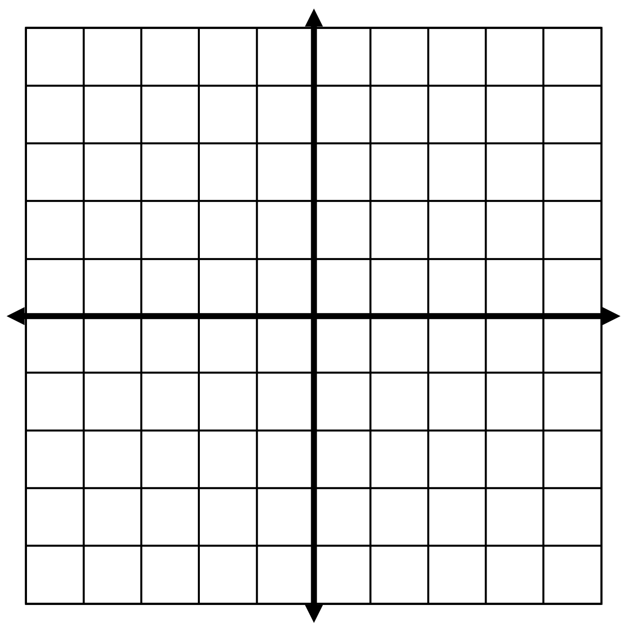
This figure is a parallelogram, and its area is,

$$A = \frac{1}{2} d_1 d_2 = \frac{1}{2} \cdot 2 \cdot 4 = 4$$

Notice that the determinant of the transform matrix is,

$$\begin{vmatrix} -1 & -1 \\ 2 & -2 \end{vmatrix} = (-1)(-2) - (2)(-1) = 4$$

We can see that a 2×2 matrix transforms the unit square into a parallelogram. Such a matrix is represents a ***linear transformation***, and the area of the resulting parallelogram is the absolute value of the determinant of the matrix.

***Exercise 4*:**	Use the linear transformation represented by the following matrix to transform the unit square. Graph the resulting parallelogram and determine its area. $\begin{bmatrix} -1 & -1 \\ 1 & 3 \end{bmatrix}$ 
Class work:	Class work Handout 2-3
Homework:	Homework Handout 2-3