Transition to College Math

Name ______________________________

Period ________

Date: Unit 2: **Matrices and Vectors** Lesson 2: **Multiplying Matrices**	Essential Question: **Suppose you have two matrices, $\mathbf{A}_{2,2}$ and $\mathbf{B}_{2,3}$. is $\mathbf{A} \times \mathbf{B}$ defined? What about $\mathbf{B} \times \mathbf{A}$? Explain.**
Standard: N.VM.8	Add, subtract, and multiply matrices of appropriate dimensions.
Learning Target: ***Elements of a matrix:***	The student will learn how to multiply matrices and to use the properties of matrix multiplication. 80% of the students will be able to perform the following multiplication: $\begin{bmatrix} -9 \\ 6 \end{bmatrix} \begin{bmatrix} -1 & -10 & 1 \end{bmatrix}$ The elements of a matrix can be identified by two subscripts. Suppose $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ This can be written, $\mathbf{A} = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}$ Clearly, $a_{1,1} = 1$, $a_{1,2} = 2$, $a_{2,1} = 3$, and $a_{2,2} = 4$, The first subscript is the row, and the second subscript is the column. For example, $b_{4,2}$ is the element of matrix $\mathbf{B}$ that is in the 4th row and the 2nd column.
Summary	

	We can use a similar notation to indicate the dimensions of a matrix. On the previous page, **A** has two rows and two columns. We would indicate that by writing, $\mathbf{A}_{2,2}$.
***Example 1*:**	If $\mathbf{C} = \begin{bmatrix} 1 & -3 & -5 \\ -2 & 6 & 4 \\ 0 & 7 & -4 \end{bmatrix}$, find $c_{2,3}$. The element that is in the 2nd row and 3rd column is 4. Therefore, $c_{2,3} = 4$. Moreover, we would indicate the dimensions of **C** as follows: $\mathbf{C}_{3,3}$.
***Exercise 1*:**	If $\mathbf{D} = \begin{bmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{bmatrix}$, find $d_{2,3}$.
***Multiplying Matrices*:**	Multiplying two matrices can best be illustrated by examining the following example: $$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1 \times 5 + 2 \times 7 & 1 \times 6 + 2 \times 8 \\ 3 \times 5 + 4 \times 7 & 3 \times 6 + 4 \times 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$ In this example, the elements in the first row of the first matrix multiply the elements in the first column of the second matrix. Then the products are added. This sum of products becomes the element in the first row and first column of the product matrix. The process continues by multiplying elements of the rows in the first matrix by elements of the columns in the second matrix and summing the products. These sums of products become the elements in the corresponding rows and columns of the product matrix.

	Let $\mathbf{A}_{m,n} \times \mathbf{B}_{n,r} = \mathbf{D}_{m,r}$ then, $$d_{j,k} = \sum_{i=1}^{n} a_{j,i} \times b_{i,k}$$
Example 2:	Multiply, $$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \times \begin{bmatrix} 7 & 8 \\ 11 & 12 \end{bmatrix}$$ $$= \begin{bmatrix} 1 \times 7 + 4 \times 11 & 1 \times 8 + 4 \times 12 \\ 2 \times 7 + 5 \times 11 & 2 \times 8 + 5 \times 12 \\ 3 \times 7 + 6 \times 11 & 3 \times 8 + 6 \times 12 \end{bmatrix}$$ $$= \begin{bmatrix} 51 & 56 \\ 69 & 76 \\ 87 & 96 \end{bmatrix}$$
Exercise 2:	Multiply, $$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \times \begin{bmatrix} 2 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

***Rules for Multiplying Matrices*:**

1. The first matrix (the one on the left) must have the same number of columns as the second matrix (the one on the right) has rows.

2. The product matrix has the same number of rows as the first matrix and the same number of columns as the second matrix.

Specifically, $\mathbf{A}_{m,n} \times \mathbf{B}_{n,r} = \mathbf{D}_{m,r}$.

***Commutative Property*:**

In the previous lesson, we found that matrices are commutative under addition. That is,

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$

What about multiplication? Is the following equation true?

$$\mathbf{A} \times \mathbf{B} = \mathbf{B} \times \mathbf{A}$$

***Example 3*:**

Let's try an example. Multiply the following:

$$\begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} \times \begin{bmatrix} -1 & -2 \\ 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1(-1) + (-2)\cdot 2 & 1(-2) + (-2)\cdot 0 \\ 0(-1) + 3\cdot 2 & 0(-2) + 3\cdot 0 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & -2 \\ 6 & 0 \end{bmatrix}$$

Now, let's interchange the matrices and multiply again:

$$\begin{bmatrix} -1 & -2 \\ 2 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} (-1)1 + (-2)0 & (-1)(-2) + (-2)3 \\ 2\cdot 1 + 0\cdot 0 & 2(-2) + 0\cdot 3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -4 \\ 2 & -4 \end{bmatrix}$$

	Clearly, $$\begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} \times \begin{bmatrix} -1 & -2 \\ 2 & 0 \end{bmatrix} \neq \begin{bmatrix} -1 & -2 \\ 2 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}$$ This counterexample demonstrates that multiplication of matrices is ***not commutative***.
***Exercise 3*:**	Multiply the following: $$\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} \times \begin{bmatrix} -1 & 1 \\ 3 & 4 \end{bmatrix}$$ $$\begin{bmatrix} -1 & 1 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}$$ Are these two products equal? Why or why not?

***Row and Column Matrices*:**	A row matrix is a matrix that has any number of columns, but only one row. The following matrix is a row matrix that has 3 columns: $\mathbf{A}_{1,3} = \begin{bmatrix} 4 & 5 & 6 \end{bmatrix}$ A column matrix is a matrix that has any number of rows, but only one column. The following matrix is a column matrix that has 3 rows: $\mathbf{B}_{3,1} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$
***Example 4*:**	Multiply the row matrix above by the column matrix above. $\mathbf{A}_{1,3} \times \mathbf{B}_{3,1} = \mathbf{D}_{1,1}$ Since $\mathbf{A}$ is 1×3 and $\mathbf{B}$ is 3×1, we expect the product matrix to be 1×1. $\begin{bmatrix} 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = [4 \cdot 1 + 5 \cdot 2 + 6 \cdot 3] = [32]$
***Exercise 4*:**	Multiply the column matrix above by the row matrix above. Specifically, find the following. First determine the dimensions of $\mathbf{D}$. Then use the formula in the box at the top of page 3 to calculate each term of $\mathbf{D}$. $\mathbf{B}_{3,1} \times \mathbf{A}_{1,3} = \mathbf{D}$

Exercise 4, continued:	
Associative Property:	As we have seen, matrix multiplication is not commutative, but it is associative. That is, if the matrix products are defined, then, $$\boxed{\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}}$$
Example 5:	Multiply, $\begin{bmatrix} -1 & 1 \\ 3 & 4 \end{bmatrix} \times \left(\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} \times \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} \right)$ $= \begin{bmatrix} -1 & 1 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} -4 & -3 \\ 13 & 1 \end{bmatrix}$ $= \begin{bmatrix} 17 & 4 \\ 40 & -5 \end{bmatrix}$ $\left(\begin{bmatrix} -1 & 1 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} \right) \times \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$ $= \begin{bmatrix} 1 & 5 \\ 11 & 6 \end{bmatrix} \times \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$ $= \begin{bmatrix} 17 & 4 \\ 40 & -5 \end{bmatrix}$

***Exercise 5*:**	Test the associative property yourself. Show that $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$ $\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 1 & 2 \end{bmatrix}$ $\mathbf{B} = \begin{bmatrix} -4 & -1 \\ 2 & 5 \end{bmatrix}$ $\mathbf{C} = \begin{bmatrix} -3 & -2 \\ -1 & 2 \end{bmatrix}$

***Distributive Property*:**	What about the distributive property? Does it apply to matrix algebra? Specifically, are the following matrix equations true? $\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$ and $(\mathbf{B} + \mathbf{C}) \times \mathbf{A} = \mathbf{B} \times \mathbf{A} + \mathbf{C} \times \mathbf{A}$
***Exercise 6*:**	Use the following matrices to show that $(\mathbf{B} + \mathbf{C}) \times \mathbf{A} = \mathbf{B} \times \mathbf{A} + \mathbf{C} \times \mathbf{A}$ is true. $\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 1 & 2 \end{bmatrix}$ $\mathbf{B} = \begin{bmatrix} -4 & -1 \\ 2 & 5 \end{bmatrix}$ $\mathbf{C} = \begin{bmatrix} -3 & -2 \\ -1 & 2 \end{bmatrix}$

Exercise 6 Continued:	
Class work:	Guided Practice handout: Matrix Multiplication
Homework:	Homework handout: Matrix Multiplication