Chapter 8.2

L'Hôpital's Rule

Objective

- Find the limits of indeterminate forms using L'Hôpital's Rule.

Learning Target

80% of the students will be able to use L'Hôpital's Rule to evaluate

$$\lim_{x\to\infty}(\ln x)^{1/x}\ .$$

Standard

F-IF.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

Overview

- Indeterminate form $0/0$
- Indeterminate forms ∞/∞, $\infty \cdot 0$, $\infty - \infty$
- Indeterminate forms 1^{∞}, 0^0, ∞^0

Indeterminate Form $0/0$

If the functions $f(x)$ and $g(x)$ are both zero when $x = a$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

cannot be found by substituting $x = a$.

The meaningless expression $0/0$ is an indeterminate form.

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$$

L'Hôpital's Rule uses our success with derivatives to find the limits of indeterminate forms.

L'Hôpital's Rule (First Form)

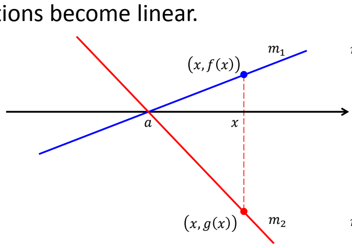
Suppose that $f(a) = g(a) = 0$, that $f'(a)$ and $g'(a)$ exist, and that $g'(a) \neq 0$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

Graphical Argument

Let's zoom in on the region $x = a$.

If we zoom in tightly enough, the differentiable functions become linear.



$$m_1 = \frac{f(x)}{x - a}$$

$$m_2 = \frac{g(x)}{x - a}$$

$$\frac{f(x)}{g(x)} = \frac{\frac{f(x)}{x-a}}{\frac{g(x)}{x-a}}$$

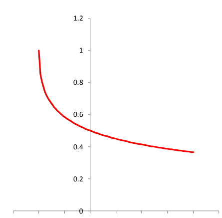
$$= \frac{m_1}{m_2}$$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{m_1}{m_2} = \frac{f'(a)}{g'(a)}$$

Example 1

Estimate the limit graphically and then use L'Hôpital's Rule to find the following limit:

$$\lim_{x \to 0} \frac{\sqrt{1+x}-1}{x}$$



$$f(x) = \sqrt{1+x} - 1$$

$$g(x) = x$$

$$f'(x) = \frac{1}{2\sqrt{1+x}}$$

$$g'(x) = 1$$

$$\lim_{x \to 0} \frac{\sqrt{1+x}-1}{x} = \lim_{x \to 0} \frac{\frac{1}{2\sqrt{1+x}}}{1} = \frac{1}{2}$$

Exercise 1

Estimate the limit graphically and then use L'Hôpital's Rule to find the following limit :

$$\lim_{x \to 1} \frac{\sqrt[3]{x} - 1}{x - 1}$$

L'Hôpital's Rule (Stronger Form)

Sometimes the new numerator and denominator are both zero.

Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

if the latter limit exists.

Example 2

Evaluate:

$$\lim_{x \to 0} \frac{\sqrt{1 + x} - 1 - x/2}{x^2}$$

Substituting $x = 0$ leads to $0/0$.

$$\lim_{x \to 0} \frac{\sqrt{1 + x} - 1 - x/2}{x^2} = \lim_{x \to 0} \frac{\frac{1}{2\sqrt{1 + x}} - \frac{1}{2}}{2x}$$

Substituting $x = 0$ once again leads to $0/0$.

Therefore, we differentiate again.

$$\lim_{x\to 0}\frac{\frac{1}{2\sqrt{1+x}}-\frac{1}{2}}{2x}=\lim_{x\to 0}\frac{-\frac{1}{4(1+x)^{3/2}}}{2}$$

Therefore,

$$\lim_{x\to 0}\frac{\sqrt{1+x}-1-x/2}{x^2}=-\frac{1}{8}$$

Exercise 2

Evaluate:

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}$$

Exploration

$$f(x)=\frac{\sin x}{x}$$

- Use L'Hôpital's Rule to find $\lim_{x\to 0} f(x)$.
- Let $y_1=\sin x$, $y_2=x$, $y_3=y_1/y_2$, $y_4={y_1}'/{y_2}'$.
- How does graphing y_3 and y_4 in the same window supports L'Hôpital's Rule.

Exploration, continued

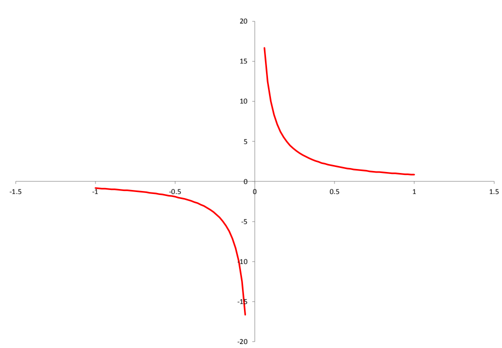
- Let $y_5 = y_3{}'$.
- Graph y_3, y_4, and y_5 in the same window.
- Make a statement about what L'Hôpital's Rule does *not* say.

Example 3, One-Sided Limits

Evaluate the following limits using L'Hôpital's Rule:

$$\lim_{x\to 0^+} \frac{\sin x}{x^2} = \lim_{x\to 0^+} \frac{\cos x}{2x} = \frac{1}{0} = \infty$$

$$\lim_{x\to 0^-} \frac{\sin x}{x^2} = \lim_{x\to 0^-} \frac{\cos x}{2x} = -\frac{1}{0} = -\infty$$



If one of the derivatives approaches zero, and the other does not, then

1. The limit is 0 if the numerator goes to 0.
2. The limit is $\pm\infty$ if the denominator goes to 0.

Exercise 3

Evaluate the following limits using L'Hôpital's Rule:

$$\lim_{x\to 0^-} \frac{\sin x}{x^3}$$

$$\lim_{x\to 0^+} \frac{\sin x}{x^3}$$

Indeterminate Form ∞/∞

If

$$\lim_{x\to\infty} f(x) = \infty$$

and

$$\lim_{x\to\infty} g(x) = \infty$$

then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

provided the limit exists.

Example 4

Identify the indeterminate form and evaluate the following limit using L'Hôpital's Rule:

$$\lim_{x \to \pi/2} \frac{\sec x}{1 + \tan x}$$

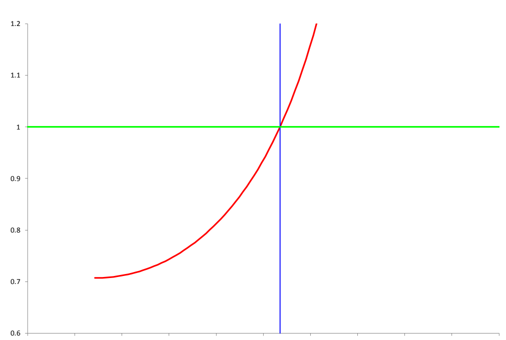
$$\lim_{x \to \pi/2} \sec x = \lim_{x \to \pi/2} (1 + \tan x) = \infty$$

Therefore, this is the ∞/∞ indeterminate form.

$$\lim_{x \to \pi/2} \frac{\sec x}{1 + \tan x} = \lim_{x \to \pi/2} \frac{\frac{d}{dx}(\sec x)}{\frac{d}{dx}(1 + \tan x)}$$

$$= \lim_{x \to \pi/2} \frac{\sec x \tan x}{\sec^2 x} = \lim_{x \to \pi/2} \frac{\tan x}{\sec x}$$

$$= \lim_{x \to \pi/2} \sin x = 1$$



Exercise 4

Identify the indeterminate form and evaluate the following limit using L'Hôpital's Rule:

$$\lim_{x \to \pi} \frac{\csc x}{1 + \cot x}$$

Support your answer graphically.

Example 5

Identify the indeterminate form and evaluate the following limit using L'Hôpital's Rule:

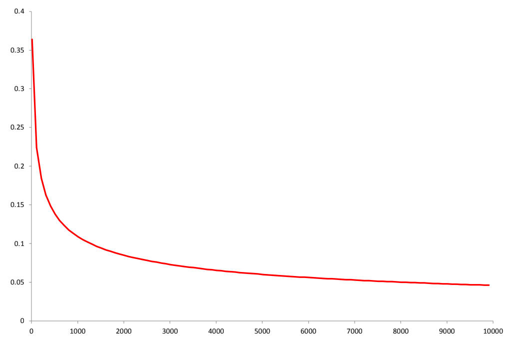
$$\lim_{x \to \infty} \frac{\ln x}{2\sqrt{x}}$$

$$\lim_{x \to \infty} \ln x = \lim_{x \to \infty} (2\sqrt{x}) = \infty$$

Therefore, this is the ∞/∞ indeterminate form.

$$\lim_{x \to \infty} \frac{\ln x}{2\sqrt{x}} = \lim_{x \to \infty} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(2\sqrt{x})}$$

$$= \lim_{x \to \infty} \frac{1/x}{1/\sqrt{x}} = \lim_{x \to \infty} \frac{1}{\sqrt{x}} = 0$$



Exercise 5

Identify the indeterminate form and evaluate the following limit using L'Hôpital's Rule:

$$\lim_{x\to\infty} \frac{5x^2 - 3x}{7x^2 + 1}$$

Support your answer graphically.

Indeterminate Form $\infty \cdot 0$

Sometimes we can use algebra to convert $\infty \cdot 0$ or $\infty - \infty$ to $0/0$ or ∞/∞.

This does not mean that $\infty \cdot 0$ is a number. Neither is $\infty - \infty$ a number.

These are descriptions of function behavior.

Example 6

Find

$$\lim_{x\to\infty}\left(x\sin\frac{1}{x}\right)$$

$$\lim_{x\to\infty} x = \infty$$

$$\lim_{x\to\infty} \sin\frac{1}{x} = 0$$

This is a $\infty \cdot 0$ indeterminate form.

Substitute

$$h = \frac{1}{x}$$

$$\lim_{x\to\infty}\left(x\sin\frac{1}{x}\right) = \lim_{h\to 0^+}\left(\frac{\sin h}{h}\right) = 1$$

Similarly

$$\lim_{x\to -\infty}\left(x\sin\frac{1}{x}\right) = \lim_{h\to 0^-}\left(\frac{\sin h}{h}\right) = 1$$

Exercise 6

Identify the indeterminate form and evaluate the following limit using L'Hôpital's Rule:

$$\lim_{x\to 0^+}(x\ln x)$$

Example 7

Find

$$\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)$$

$$\lim_{x\to 1}\frac{1}{\ln x}=\infty$$

$$\lim_{x\to 1}\frac{1}{x-1}=\infty$$

This is a $\infty-\infty$ indeterminate form.

Find a common denominator and combine the fractions.

$$\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)=\lim_{x\to 1}\left(\frac{x-1-\ln x}{(x-1)\ln x}\right)$$

$$\lim_{x\to 1}(x-1-\ln x)=0$$

$$\lim_{x\to 1}\big((x-1)\ln x\big)=0$$

We have converted the limit to a $0/0$ indeterminate form, and we can apply L'Hôpital's Rule.

$$\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)=\lim_{x\to 1}\left(\frac{x-1-\ln x}{(x-1)\ln x}\right)$$

$$=\lim_{x\to 1}\frac{\frac{d}{dx}(x-1-\ln x)}{\frac{d}{dx}\big((x-1)\ln x\big)}$$

$$\lim_{x\to 1}\left(\frac{1-{}^{1}\!/_{x}}{\frac{(x-1)}{x}+\ln x}\right)$$

Still $0/0$

$$\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)=\lim_{x\to 1}\left(\frac{x-1}{x-1+x\ln x}\right)$$

$$=\lim_{x\to 1}\left(\frac{\frac{d}{dx}(x-1)}{\frac{d}{dx}(x-1+x\ln x)}\right)$$

$$=\lim_{x\to 1}\left(\frac{1}{1+\ln x+{}^{x}/_{x}}\right)$$

$$=\lim_{x\to 1}\left(\frac{1}{2+\ln x}\right)=\frac{1}{2}$$

Exercise 7

Identify the indeterminate form and evaluate the following limit using L'Hôpital's Rule:

$$\lim_{x\to 0^+}(\csc x-\cot x+\cos x)$$

Indeterminate Forms 1^∞, 0^0, ∞^0

Limits of these forms can, sometimes, be evaluated by first taking the logarithm and use L'Hôpital's Rule to evaluate the limit. Then take the exponential to find the original function's behavior.

$$\lim_{x\to a}\ln f(x)=L\Rightarrow\lim_{x\to a}f(x)=\lim_{x\to a}e^{\ln f(x)}=e^{L}$$

Here a can be finite or infinite.

Example 8: 1^∞

Find

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$$

Let $f(x) = (1+1/x)^x$

$$\ln f(x) = \ln\left(1+\frac{1}{x}\right)^x = x\ln\left(1+\frac{1}{x}\right)$$

$$\ln f(x) = \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$$

$$\lim_{x\to\infty}\ln f(x) = \lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$$

This is a $0/0$ indeterminate form. Using L'Hôpital's Rule,

$$\lim_{x\to\infty}\ln f(x) = \lim_{x\to\infty}\frac{\frac{d}{dx}\left(\ln\left(1+\frac{1}{x}\right)\right)}{\frac{d}{dx}\left(\frac{1}{x}\right)}$$

$$= \lim_{x\to\infty}\frac{\frac{1}{1+\frac{1}{x}}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty}\frac{1}{1+\frac{1}{x}} = 1$$

Finally

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} f(x)$$

$$= \lim_{x\to\infty} e^{\ln f(x)} = e^1 = e$$

Exercise 8

Identify the indeterminate form and evaluate the following limit using L'Hôpital's Rule:

$$\lim_{x\to 1} x^{1/(x-1)}$$

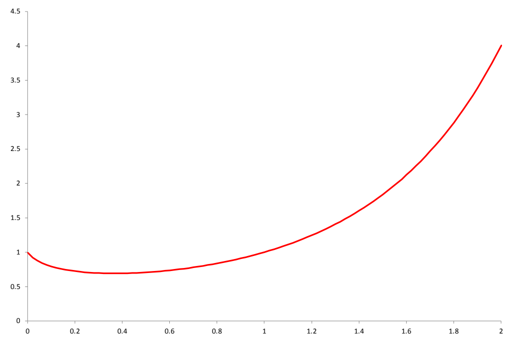
Example 9: 0^0

Determine if the following limit exists, and find it if it does.

$$\lim_{x\to 0^+} x^x$$

This is a 0^0 indeterminate form.

Let's graph $f(x) = x^x$.



Taking the logarithm,

$$\ln f(x) = \ln(x^x) = x \ln x = \frac{\ln x}{1/x}$$

$$\lim_{x \to 0^+} \ln f(x) = \lim_{x \to 0^+} \frac{\ln x}{1/x}$$

This is a $-\infty/\infty$ indeterminate form. Using L'Hôpital's Rule,

$$\lim_{x \to 0^+} \ln f(x) = \lim_{x \to 0^+} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}\left(\frac{1}{x}\right)}$$

$$\lim_{x \to 0^+} \ln f(x) = \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$$

$$= \lim_{x \to 0^+} (-x) = 0$$

Finally

$$\lim_{x \to 0^+} x^x = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} e^{\ln f(x)} = e^0 = 1$$

Exercise 9

Identify the indeterminate form and evaluate the following limit using L'Hôpital's Rule:

$$\lim_{x\to 1}(x^2 - 2x + 1)^{x-1}$$

Example 10: ∞^0

Find the limit.

$$\lim_{x\to\infty} x^{1/x}$$

This is an ∞^0 indeterminate form.

Let

$$f(x) = x^{1/x}$$

Taking the logarithm,

$$\ln f(x) = \frac{\ln x}{x}$$

Using L'Hôpital's Rule,

$$\lim_{x\to\infty} \ln f(x) = \lim_{x\to\infty} \frac{\ln x}{x}$$

This is an ∞/∞ indeterminate form; therefore, we differentiate,

$$\lim_{x\to\infty} \ln f(x) = \lim_{x\to\infty} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}x} = \lim_{x\to\infty} \frac{1/x}{1}$$

$$\lim_{x\to\infty} \ln f(x) = \lim_{x\to\infty} \frac{1}{x} = 0$$

Finally

$$\lim_{x\to\infty} x^x = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{\ln f(x)} = e^0 = 1$$

Exercise 10

Identify the indeterminate form and evaluate the following limit using L'Hôpital's Rule:

$$\lim_{x\to 0^+} \left(1 + \frac{1}{x}\right)^x$$

Homework

P 450: 3-60 multiples of 3