Chapter 6.2

Antidifferentiation by substitution

Objective

- Compute indefinite and definite integrals by the method of substitution.

Learning Target

80% of the students will be able to use substitution to evaluate

$$\int_{-\pi}^{\pi} \frac{\cos x}{\sqrt{4 + 3\sin x}}\,dx.$$

Standard

F-IF.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

Overview

- Indefinite integrals
- Leibniz Notation
- Substitution in Indefinite Integrals
- Substitution in Definite Integrals

Indefinite Integral

The family of all antiderivatives of a function $f(x)$ is the **indefinite integral of f with respect to x** and is denoted

$$\int f(x)\,dx.$$

If F is any function such that $F'(x) = f(x)$ then

$$\int f(x)\,dx = F(x) + C,$$

where C is an arbitrary constant called the **constant of integration**.

Integration By Substitution

- The basic idea behind integration by substitution is to find a way of reversing the chain rule.
- $\int 2x\cos(x^2)dx$
- Guess the antiderivative to be $\sin(x^2)$
- Check by differentiating:
- $\frac{dy}{dx}(\sin(x^2)) = \cos(x^2)\cdot 2x$ (correct)

Integration by Substitution Method

We need a formal method – guessing is unreliable.

1. Look for an expression for u, so that du is similar to the rest of the integrand.
2. Find du.
3. Substitute in u and du, and integrate.
4. Substitute the expression for u back into the result.

Example 1

Use substitution to evaluate

$$\int 2x\cos x^2\,dx$$

$$u = x^2,\ du = 2x\,dx$$

$$\int 2x\cos(x^2)\,du = \sin u + C = \sin(x^2) + C$$

Exercise 1

Use substitution to evaluate

$$\int \frac{dt}{t^2 + 1}$$

Example 2

Typically, look for the expression for u inside a radical or as a base for an exponential expression.

$$\int \frac{15x^2}{\sqrt{1 + 5x^3}} dx$$

$$u = 1 + 5x^3, \; du = 15x^2 dx$$

$$\int \frac{1}{\sqrt{u}} du = \int u^{-\frac{1}{2}} du = 2\sqrt{u} + C$$

$$= 2\sqrt{1 + 5x^3} + C$$

Exercise 2

Use substitution to evaluate

$$\int \frac{3x^2}{\sqrt{1 + x^3}} dx$$

Example 3

Evaluate

$$\int \sin x\, e^{\cos x}\, dx.$$

Let $u = \cos x$. Then $du/dx = -\sin x$. Therefore, $du = -\sin x\, dx$.

$$\int \sin x\, e^{\cos x}\, dx = -\int (-\sin x) e^{\cos x}\, dx$$

$$= -e^{u} + C$$

$$= -e^{\cos x} + C$$

Exercise 3

Use substitution to evaluate

$$\int \sqrt{\tan x}\, \sec^2 x\, dx$$

Multiplying and Dividing By A Constant

Sometimes, the expression for du doesn't match the rest of the integrand exactly. It is "off by a constant". In this case, we can multiply and divide by the constant and then substitute u and du.

$$\int x^2(x^3+5)^4\,dx$$

$$u = x^3+5, du = 3x^2dx$$

We need $3x^2dx$ in the integrand but we have only x^2dx, so we multiply and divide by 3.

$$\int \frac{3x^2(x^3+5)^4}{3}dx = \frac{1}{3}\int u^4\,du = \frac{1}{3}\left(\frac{u^5}{5}\right)+C$$

$$= \frac{1}{3}\frac{(x^3+5)^5}{5}+C = \frac{(x^3+5)^5}{15}+C$$

Example 4

$$\int 7x^2(x^3+5)^4\,dx$$

Move the 7 outside the integrand first.

$$7\int x^2(x^3+5)^4\,dx$$

$$u = x^3+5, du = 3x^2dx$$

Multiply and divide by 3.

$$\frac{7}{3}\int 3x^2(x^3+5)^4\,dx$$

$$\frac{7}{3}\int u^4\,du = \frac{7}{3}\left(\frac{u^5}{5}\right)+C = \frac{7u^5}{15}+C$$

$$= \frac{7(x^3+5)^5}{15}+C$$

Exercise 4

Use substitution to evaluate

$$\int \frac{x^3}{\sqrt{1+x^4}}\,dx$$

Example 5

"Off By a Factor of an expression involving x"

Sometimes the rest of the integrand is "off by a factor of an expression involving x".

$$\int x\sqrt{x-1}\,dx$$

$$u = x - 1$$
$$du = dx$$

We have $x\,dx$ in the integrand but we only want dx.

To rewrite the integrand in terms of u, solve for x:

$x = u + 1$

Substitute $u + 1$ in for x:

$$\int (u+1)\sqrt{u}\,du = \int (u+1)\left(u^{\frac{1}{2}}\right)du$$

$$= \int u^{\frac{3}{2}} + u^{\frac{1}{2}}\,du$$

$$= \frac{2}{5}(u)^{\frac{5}{2}} + \frac{2}{3}(u)^{\frac{3}{2}} + C$$

$$= \frac{2}{5}(x-1)^{\frac{5}{2}} + \frac{2}{3}(x-1)^{\frac{3}{2}} + C$$

Exercise 5

Use substitution to evaluate

$$\int x\sqrt[3]{x+2}\,dx$$

Example 6
Substitution in Definite Integrals

When evaluating a definite integral after using substitution, we have the option of changing the limits of integration in terms of u.

$$\int_{-2}^{1} x\sqrt{x+3}\,dx$$

$$u = x+3, x = u-3, du = dx$$

Change the limits of integration:

$$u = (1)+3 = 4; u = (-2)+3 = 1$$

$$\int_{1}^{4} (u-3)\left(u^{\frac{1}{2}}\right) du = \int_{1}^{4} u^{\frac{3}{2}} - 3u^{\frac{1}{2}} du$$

$$= \left(\frac{2}{5}u^{\frac{5}{2}} - 2u^{\frac{3}{2}}\right)\Big|_{1}^{4}$$

$$= \left(\frac{2}{5}(4)^{\frac{5}{2}} - 2(4)^{\frac{3}{2}}\right) - \left(\frac{2}{5}(1)^{\frac{5}{2}} - 2(1)^{\frac{3}{2}}\right) = -\frac{8}{5}$$

Exercise 6

Use substitution to evaluate

$$\int_0^3 \sqrt{y+1}\,dy$$

Example 7
Absolute Value

$$\int_0^1 \frac{x}{x^2-4}\,dx$$

Let $u = x^2 - 4$. Then $du = 2xdx$.

Moreover, $u(0) = -4$, and $u(1) = -3$.

$$\begin{aligned}\int_0^1 \frac{x}{x^2-4}\,dx &= \frac{1}{2}\int_0^1 \frac{2x}{x^2-4}\,dx \\ &= \frac{1}{2}\int_{-4}^{-3} \frac{du}{u} \\ &= \frac{1}{2}\ln|u|\,\Big|_{-4}^{-3} \\ &= \frac{1}{2}(\ln 3 - \ln 4) \\ &= \frac{1}{2}\ln\left(\frac{3}{4}\right)\end{aligned}$$

Properties of Indefinite Integrals

For any constant k

$$\int kf(x)\,dx = k\int f(x)\,dx$$

$$\int \big(f(x) \pm g(x)\big) = \int f(c)\,dx \pm \int g(x)\,dx$$

Power Formulas

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C$$

when $n \neq -1$

$$\int u^{-1}\,du = \int \frac{1}{u}du = \ln|u| + C$$

Trigonometric Formulas

$$\int \cos u\,du = \sin u + C$$

$$\int \sin u\,du = -\cos u + C$$

$$\int \sec^2 u\,du = \tan u + C$$

$$\int \csc^2 u\,du = -\cot u + C$$

$$\int \sec u \tan u\,du$$

$$= \sec u + C$$

$$\int \csc u \cot u\,du$$

$$= -\csc u + C$$

Exponential and Logarithmic Formulas

$$\int e^u\,du = e^u + C$$

$$\int a^u\,du = \frac{a^u}{\ln a} + C$$

$$\int \ln u\,du = u\ln u - u + C$$

$$\int \log_a u\,du = \int \frac{\ln u}{\ln a}\,du = \frac{u\ln u - u}{\ln a} + C$$

We have technology to easily evaluate definite integrals that was not available only a few years ago. Therefore, evaluating indefinite integrals has taken on less importance. Substitution is far less important than it was before the introduction of modern technology. As a result, we covered only a fraction of the substitution techniques that students routinely studied in the past.

Homework

P 337: 1-11 odd, 15-45 multiples of 3, 47, 49, 52

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