Chapter 6.1

Slope Fields and Euler's Method

Objective

- Construct antiderivatives using the Fundamental Theorem of Calculus.
- Solve initial problems in the form
$$\frac{dy}{dx} = f(x), y_0 = f(x_0)$$
- Construct slope fields using technology and interpret slope fields as visualizations of different equations.
- Use Euler's Method for graphing a solution to an initial value problem.

Learning Target

80% of the students will be able to solve the differential equation
$$\frac{dy}{dx} = -\frac{1}{x^2} - \frac{3}{x^4} + 12$$
Given the initial conditions
$$y = 3$$
when
$$x = 1.$$

Standard

F-IF.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

Overview

- Differential Equations
- Slope Fields
- Euler's Method

Differential Equations

An equation containing a derivative is a ***differential equation***. The ***order*** of a differential equation is the highest derivative in the equation.

Example 1

Find all the functions y that satisfy

$$\frac{dy}{dx} = \sec^2 x + 2x + 5.$$

The solution can be any antiderivative of $\sec^2 x + 2x + 5$.

Specifically any function of the form $y = \tan x + x^2 + 5x + C$.

This family of functions is the ***general solution*** of the differential equation.

Exercise 1

Find all the functions y that satisfy

$$\frac{dy}{dx} = \sec x \tan x - e^x.$$

Initial Value Problem

- We cannot find a unique solution to a differential equation without more information.
- If the general solution of a first-order differential equation is continuous, we need only the value of the function at a single point.
- This is called an initial condition.
- With this information, we have a unique solution, called a ***particular solution***.

Example 2

Find the particular solution to the equation

$$\frac{dy}{dx} = e^x - 6x^2$$

whose graph passes through the point $(1, 0)$.
The general solution is $y = e^x - 2x^3 + C$.
Applying the initial conditions,

$$0 = e^1 - 2 \cdot 1^3 + C$$
$$C = 2 - e$$

The particular solution is

$$y = e^x - 2x^3 + 2 - e.$$

Exercise 2

Find the particular solution to the equation

$$\frac{du}{dx} = 7x^6 - 3x^2 + 5$$

and $u = 1$ when $x = 1$.

Discontinuities

- An initial condition determines a particular solution by requiring that solution curve passes through a particular point.
- If the curve is discontinuous, the initial condition defines the particular solution only on the continuous portion containing the initial value point.
- The domain must be specified.

Example 3

Find the particular solution to the equation
$$\frac{dy}{dx} = 2x - \sec^2 x$$
whose graph passes through the point $(0, 3)$.
The general solution is $y = x^2 - \tan x + C$.
Applying the initial conditions,
$$3 = 0^2 - \tan 0 + C$$
$$C = 3$$
The particular solution is
$$y = x^2 - \tan x + 3.$$

$\tan x$ is undefined at $x = \pi/2 + n\pi$.

The function is continuous on the open interval $(-\pi/2, \pi/2)$, which contains $x = 0$.

This particular solution is defined only over the domain

$$-\frac{\pi}{2} < x < \frac{\pi}{2}.$$

Exercise 3

Find the particular solution to the equation
$$\frac{dv}{dt} = 4 \sec t \tan t + e^t + 6t$$
and $v = 5$ when $t = 0$.

Fundamental Theorem

- Sometimes we cannot find an antiderivative.
- We can still use the Fundamental Theorem of calculus.
- Modern technology aids us.

Example 4

Find the solution to the differential equation

$$f'(x) = e^{-x^2},$$

and $f(7) = 3$.

$$f(x) = \int_7^x e^{-t^2}\,dt + 3$$

Use a graphing calculator to find $f(-2)$.

$$\int_7^{-2} (e^{-T^2})\,dT + 3$$

1.231691684

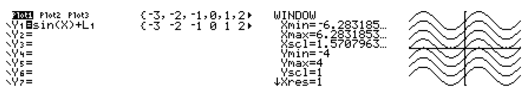
Exercise 4

Find the particular solution to the equation $\frac{du}{dx} = \sqrt{2 + \cos x}$ and $u = -3$ when $x = 0$.

Example 5

Graph the family of functions that solve the differential equation

$$\frac{dy}{dx} = \cos x$$
$$y = \sin x + C$$



Exercise 5

Graph the family of functions that solve the differential equation

$$\frac{dy}{dx} = \sin x$$

Slope Field

We do not have to solve the differential equation.

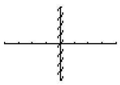
Graph a small, linearized segment.

This approximates the solution of the differential equation.

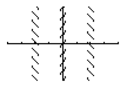
The next example illustrates this process.

Example 6

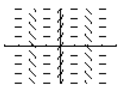
$\cos 0 = 1$



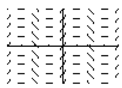
$\cos(\pm\pi) = -1$



$\cos\left(\frac{2n-1}{2}\pi\right) = 0$



$\cos(2n\pi) = 1$

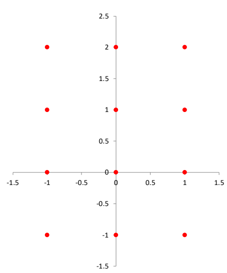


Exercise 6

Draw a slope field for $\frac{dy}{dx} = x.$

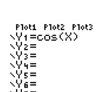
Draw tiny segments through the lattice points.

Use slope analysis, not your graphing calculator.



Using SLOPEF

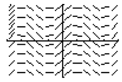
Enter the differential equation into the equation editor.



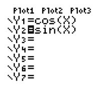
Select SLOPEF from the program list.

prgmSLOPEF

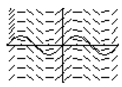
Execute SLOPEF.



Add $y = \sin x$ to the equation editor.



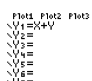
Graph it.

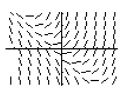


Example 7

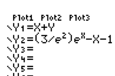
Generating a slope field when you cannot define the derivative analytically.

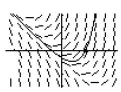
$$\frac{dy}{dx} = x + y$$





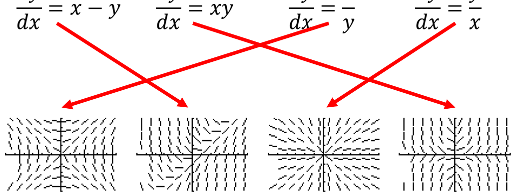
$$y = Ce^{x} - x - 1$$





Use slope analysis to match each of the following differential equations with one of the slope fields shown.

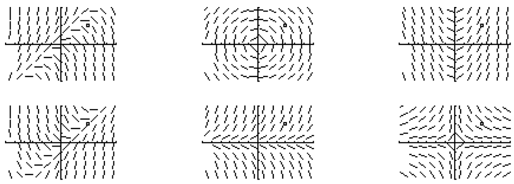
$\frac{dy}{dx} = x - y$ $\quad \frac{dy}{dx} = xy$ $\quad \frac{dy}{dx} = \frac{x}{y}$ $\quad \frac{dy}{dx} = \frac{y}{x}$



Exercise 7

Match the slope field for $\frac{dy}{dx} = x$.

Then sketch the particular solution that passes through the indicated point.



Euler's Method

In the first part of Example 7, we generated a slope field and drew a curve through a point that matched the slope field to generate a particular solution.

We could start with the point and draw small, linear segments with the correct slope.

In this way we could build an approximation of the particular solution.

This is **Euler's Method**.

Euler's Method

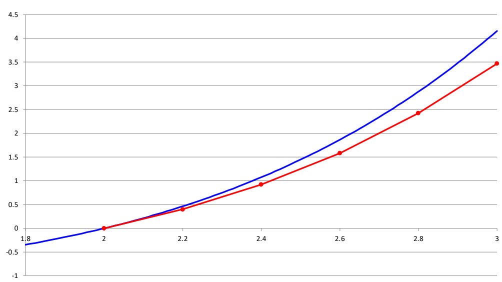
1. Begin at the point (x, y).
2. Find the slope $\frac{dy}{dx}$ at the point.
3. Find $\Delta y = \frac{dy}{dx}\Delta x$, where Δx is a small increment.
4. Graph a linear segment between (x, y) and $(x + \Delta x, y + \Delta y)$.
5. Use $(x + \Delta x, y + \Delta y)$ as the new point, return to step 2.
6. Repeat this process as many times as necessary.
7. Go to the left using a negative Δx.

Example 8

Us Euler's method to approximate the particular solution for $dy/dx = x + y$ that passes through the point $(2, 0)$.

x	y	dy/dx	dy
2	0	2	0.4
2.2	0.4	2.6	0.52
2.4	0.92	3.32	0.664
2.6	1.584	4.184	0.8368
2.8	2.4208	5.2208	1.04416
3	3.46496	6.46496	1.292992

Euler's Method



Exercise 8

Use Euler's Method with increments of $\Delta x = 0.01$ to approximate the value of y when $x = 1.3$ of

$$\frac{dy}{dx} = x - 1$$

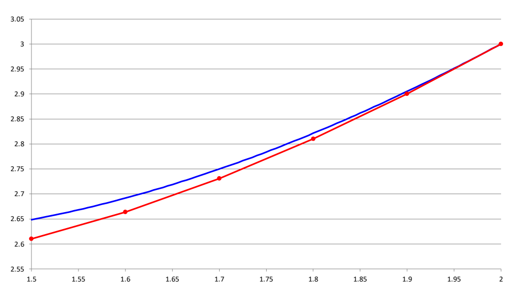
if $y = 2$ when $x = 1$.

Example 9

If $dy/dx = 2x - y$ and $y = 3$ when $x = 2$, use Euler's Method with five equal steps to approximate y when $x = 1.5$.

x	y	dy/dx	dy
2	3	1	-0.1
1.9	2.9	0.9	-0.09
1.8	2.81	0.79	-0.079
1.7	2.731	0.669	-0.0669
1.6	2.6641	0.5359	-0.05359
1.5	2.61051	0.38949	-0.03895

Euler's Method



Exercise 9

Use Euler's method with increments of $\Delta x = 0.1$ to approximate the value of y when $x = 1.3$.

$$\frac{dy}{dx} = x - 1$$

and

$$y = 1$$

when

$$x = 2$$

Example 10

If

$$\frac{dy}{dx} = 2x - y$$

and

$$y = 3$$

when

$$x = 2$$

use Euler's method with five equal steps to approximate y when $x = 1.5$.

Starting at $x = 2$, we need five equal steps of $\Delta x = -0.1$.

(x, y)	$\frac{dy}{dx} = 2x - y$	Δx	$\Delta y = \frac{dy}{dx}\Delta x$	$(x + \Delta x, y + \Delta y)$
(2,3)	1	-0.1	-0.1	(1.9,2.9)
(1.9,2.9)	0.9	-0.1	-0.09	(1.8,2.81)
(1.8,2.81)	0.79	-0.1	-0.079	(1.7,2.731)
(1.7,2.731)	0.669	-0.1	-0.0669	(1.6,2.6641)
(1.6,2.6641)	0.5359	-0.1	-0.05359	(1.5,2.61051)

The actual value is ~2.649. (~1.4% error)

Exercise 10

Use Euler's method with increments of $\Delta x = -0.1$ to approximate the value of y when $x = 1.7$.

$$\frac{dy}{dx} = x - y$$

and

$$y = 2$$

when

$$x = 2$$

Homework

P 327: 3-24 X 3, 25-28, 30-51 X 3 52, 53