# Chapter 5.5

Trapezoidal Rule

## Objective

- Approximate the definite integral by using the Trapezoidal Rule and by using Simpson's Rule, and estimate the error in using the Trapezoidal and Simpson's Rules.

## Learning Target

80% of the students will be able to to ust the Trapezoidal Rule to estimate

$$\int_1^2 x^2\,dx.$$

## Standard

F-IF.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

## Overview

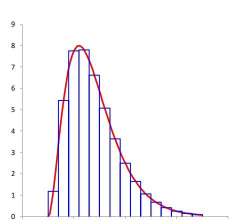
- Trapezoidal Approximations
- Other Algorithms
- Error Analysis

## Approximating Definite Integrals

We started this chapter by using sums of rectangles to approximate definite integrals.

We found that the middle approximation was often the best.

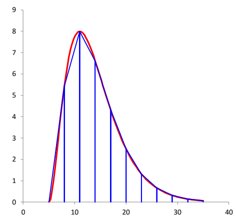
Is there anything better?



## Trapezoidal Approximations

Instead of using rectangles, we can use trapezoids.

This is called a trapezoidal approximation, or the **Trapezoidal Rule**.




## Trapezoidal Rule

Subdivide the interval into $n$ segments each of length $\Delta x$. Then approximate the integral as a sum of $n$ trapezoids of width $\Delta x$.

The $i^{th}$ trapezoid has area

$$A_i = \frac{1}{2}\Delta x\big(f(x_{i-1}) + f(x_i)\big),$$

and the definite integral is approximately

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} A_i = \frac{\Delta x}{2}\sum_{i=1}^{n}\big(f(x_{i-1}) + f(x_i)\big).$$

## Trapezoidal Rule

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\left(\sum_{i=0}^{n-1} f(x_i) + \sum_{i=1}^{n} f(x_i)\right)$$

$$= \frac{\big(\Delta x\big(f(x_0) + f(x_1) + \cdots + f(x_{n-1})\big)\big)}{2} + \frac{\big(\Delta x\big(f(x_1) + f(x_2) + \cdots + f(x_n)\big)\big)}{2}$$

$$\int_a^b f(x)\,dx \approx T =$$
$$\frac{\Delta x}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-2} + 2y_{n-1} + y_n)$$

where $[a, b]$ is partitioned into $n$ equal subintervals, each with length

$$\Delta x = \frac{b-a}{n}$$
$$T = \frac{LRAM_n + RRAM_n}{2}$$

where $LRAM_n$ and $RRAM_n$ are the left and right Riemann sums for $n$ partitions for $f(x)$, respectively.

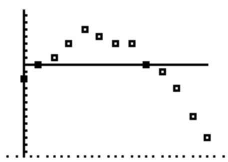
## Example 1

An observer measures the outside temperature every hour from noon until midnight, recording the temperatures in the following table. What was the average table? Use the Trapezoidal Rule to approximate $\int_0^{12} T(t)\,dt$.

| Time | N | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | M |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Temp | 63 | 65 | 66 | 68 | 70 | 69 | 68 | 68 | 65 | 64 | 62 | 58 | 55 |

$$av(f) = \frac{1}{b-a}\int_a^b f(x)\,dx$$
$$\int_a^b f(x)\,dx \approx T$$
$$= \frac{\Delta x}{2}(y_0 + 2y_1 + \cdots + 2y_{n-1} + y_n)$$
$$= \frac{1}{2}(63 + 2\cdot 65 + 2\cdot 66 + 2\cdot 68 + 2\cdot 70 + 2 \cdot 69 + 2\cdot 68 + 2\cdot 68 + 2\cdot 65 + 2\cdot 64 + 2 \cdot 62 + 2\cdot 58 + 55) = \frac{1564}{2} = 782$$

$$av(f) \approx \frac{782}{12} \approx 65.167°$$



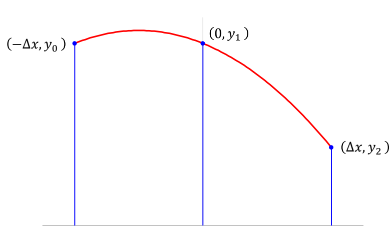
## Exercise 1

Use the function values in the following table to approximate the integral

$$\int_2^8 f(x)\,dx$$

| $x$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 16 | 19 | 17 | 14 | 13 | 16 | 20 |

## Area Under a Parabolic Arc




$$A_p = \frac{\Delta x}{3}(y_0 + 4y_1 + y_2)$$

## Simpson's Rule

Approximate

$$\int_a^b f(x)\,dx$$

Divide the interval $[a, b]$ into an *even* number of subintervals of equal length

$$\Delta x = \frac{b-a}{n}$$

## Simpson's Rule

$$\int_a^b f(x)\,dx \approx S$$

$$S = \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots$$
$$+2y_{n-4} + 4y_{n-3} + 2y_{n-2} + 4y_{n-1} + y_n)$$

## Example 2

Use Simpson's Rule with $n = 4$ to approximate

$$\int_0^2 5x^4\,dx.$$

| $i$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $x_i$ | 0 | $\frac{1}{2}$ | 1 | $\frac{3}{2}$ | 2 |
| $y_i$ | 0 | $\frac{5}{16}$ | 5 | $\frac{405}{16}$ | 80 |

$$s = \frac{0.5}{3}\left(0 + 4 \cdot \frac{5}{16} + 2 \cdot 5 + 4 \cdot \frac{405}{16} + 80\right)$$

$$\approx 32.083$$

$$\int_0^2 (5x^4)\,dx$$

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## Exercise 2

Use Simpson's Rule with $n = 4$ to approximate

$$\int_0^4 \sqrt{x}\, dx.$$

## How accurate is our approximation?

**Trapezoidal Rule:**

$$|E_T| \leq \frac{b-a}{12}(\Delta x)^2 M_{f''}$$

$M_{f''}$ is the maximum of the second derivative.

**Simpson's Rule**

$$|E_S| \leq \frac{b-a}{180}(\Delta x)^4 M_{f^{(4)}}$$

$M_{f^{(4)}}$ is the maximum of the fourth derivative.

## Example 3

Use the Trapezoidal Rule with $n = 4$ to approximate

$$\int_0^2 5x^4\,dx.$$

| $i$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $x_i$ | 0 | $\frac{1}{2}$ | 1 | $\frac{3}{2}$ | 2 |
| $y_i$ | 0 | $\frac{5}{16}$ | 5 | $\frac{405}{16}$ | 80 |

$$s = \frac{0.5}{2}\left(0 + 2\cdot\frac{5}{16} + 2\cdot 5 + 2\cdot\frac{405}{16} + 80\right)$$

$$= 35.3125$$

$$\int_0^2 (5x^4)dX$$

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## Error Analysis

$$f'' = 60x^2$$
$$M_{f''} = f''(2) = 240$$
$$|E_T| \le \frac{2-0}{12}\left(\frac{1}{2}\right)^2 \cdot 240 = 10$$
$$f^{(4)} = 120$$
$$|E_S| \le \frac{2-0}{180}\left(\frac{1}{2}\right)^4 \cdot 120 \approx 0.083$$

## Homework

P 312: 1, 4, 7, 9, 10, 13, 16, 19, 20, 23, 26-28, 39