Chapter 5.4

Fundamental Theorem of Calculus

Objective

- Apply the Fundamental Theorem of Calculus.
- Understand the relationship between the derivative and definite integral as expressed in both parts of the Fundamental Theorem of Calculus.

Learning Target

80% of the students will be able to evaluate $\int_0^2 (-32t + 16)\, dt$.

Standard

F-IF.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

Overview

- Fundamental Theorem, Part 1
- Graphing the Function $\int_a^x f(t)\,dt$
- Fundamental Theorem, Part 2
- Area Connection
- Analyzing Antiderivatives Graphically

Let the function $F(x)$ be defined on the closed interval $[a, b]$,

$$F(x) = \int_a^x f(t)\,dt$$

Then,

$$\frac{dF}{dx} = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h\to 0} \frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h}$$

$$F(x) = \lim_{h \to 0} \frac{\int_x^{x+h} f(t)\,dt}{h}$$
$$= \lim_{h \to 0} \left[\frac{1}{h}\int_x^{x+h} f(t)\,dt\right]$$

The expression in brackets is the average value of f from x to $x + h$. The Mean Value Theorem for Integrals assures us that f takes on the average value at least once in the interval. (f is continuous on the closed interval $[x, x + h]$.)

$$\frac{1}{h}\int_x^{x+h} f(t)\,dt = c$$

for some c between x and $x + h$. Therefore,

$$\frac{dF}{dx} = \lim_{h \to 0} f(c)$$

As h approaches 0, c comes closer to x.

Finally, in the limit $h = 0$, $c = x$.

$$\lim_{h \to 0} f(c) = f(x)$$

Summary

$$\frac{dF}{dx} = \lim_{h \to 0} \frac{F(x + h) - F(x)}{h}$$
$$= \lim_{h \to 0} \frac{\int_x^{x+h} f(t)\,dt}{h}$$
$$= \lim_{h \to 0} f(c)$$
$$= f(x)$$

The Fundamental Theorem of Calculus, part 1

If a function f is continuous on the closed interval $[a, b]$, then the function

$$F(x) = \int_a^x f(t)\,dt$$

has a derivative at every point in $[a, b]$, and

$$\frac{dF}{dx} = \frac{d}{dx}\int_a^x f(t)\,dt.$$

Example 1

- Find

$$\frac{d}{dx}\int_{-\pi}^x \cos t\,dt = \cos x$$

$$\frac{d}{dx}\int_0^x \frac{1}{1+t^2}\,dt = \frac{1}{1+x^2}$$

Exercise 1

- Find

$$\frac{d}{dx}\int_0^x (t^3 - t)^5\,dt$$

Example 2

Use the Chain Rule to find

$$\frac{d}{dx}\int_1^{x^2} \cos t \, dt$$

Since the upper limit is not x, we make the substitution of $u = x^2$.

The chain rule is

$$\frac{df(u)}{dx} = \frac{df(u)}{du} \cdot \frac{du}{dx}$$

Therefore,

$$\frac{d}{dx}\int_1^{x^2} \cos t \, dt = \left(\frac{d}{dx}\int_1^{x^2} \cos t \, dt\right) \cdot \frac{du}{dx}$$

$$= \cos u \cdot \frac{du}{dx}$$

$$= \cos x^2 \cdot 2x$$

$$= 2x \cos x^2$$

Exercise 2

Use the Chain Rule to find

$$\frac{d}{dx}\int_0^{x^2} e^{t^2} \, dt$$

Example 3

Use the rules if integration to find

$$\frac{d}{dx}\int_x^5 3t \sin t \, dt$$
$$= \frac{d}{dx}\left(-\int_5^x 3t \sin t \, dt\right)$$
$$= -\frac{d}{dx}\int_5^x 3t \sin t \, dt$$
$$= -3x \sin x$$

$$\frac{d}{dx}\int_{2x}^{x^2} \frac{1}{2+e^t} dt$$
$$= \frac{d}{dx}\left(\int_0^{x^2} \frac{1}{2+e^t} dt - \int_0^{2x} \frac{1}{2+e^t} dt\right)$$
$$= \frac{1}{2+e^{x^2}} \frac{d}{dx}(x^2) - \frac{1}{2+e^{2x}} \frac{d}{dx}(2x)$$
$$= \frac{2x}{2+e^{x^2}} - \frac{2}{2+e^{2x}}$$

Exercise 3

Use the rules for integration to find

$$\frac{d}{dx}\int_{x^2}^{x^3} \cos(2t) \, dt$$

Example 4

Find a function $y = f(x)$ with derivative

$$\frac{dy}{dx} = \tan x$$

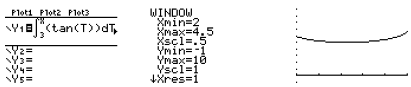
and $f(3) = 5$.

The Fundamental Theorem of Calculus:

$$y = \int_3^x \tan x \, dx$$

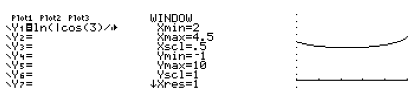
Since $f(3) = 5$, the function that has the correct derivative and the correct value at $x = 3$ is

$$y = \int_3^x \tan t \, dt + 5$$



The analytical solution with $f(3) = 5$ that has the correct derivative and the correct value at $x = 3$ is

$$y = \ln\left|\frac{\cos 3}{\cos x}\right| + 5$$



Exercise 4

Use the Fundamental Theorem of Calculus to find the function $y = f(x)$ with

$$\frac{dy}{dx} = \cos^2 5x$$

with $y = -2$ when $x = 7$.

Finding Definite Integrals

If a function f is continuous on the closed interval $[a, b]$ and F is an antiderivative of f on $[a, b]$, then by the Fundamental Theorem of Calculus,

$$\frac{dy}{dx}\int_a^x f(t)\,dt = \frac{d}{dx}F(x)$$

and

$$\int_a^x f(t)\,dt = F(x) - F(a)$$

$$\int_a^b f(x)\,dx = \int_a^x f(t)\,dt + \int_x^b f(t)\,dt$$

$$= \int_a^x f(t)\,dt - \int_b^x f(t)\,dt$$

$$= F(x) - F(a) - \big(F(x) - F(b)\big)$$

$$= F(x) - F(x) + F(b) - F(a)$$

$$= F(b) - F(a)$$

The Fundamental Theorem of Calculus, part 2

If a function f is continuous on the closed interval $[a, b]$ and F is an antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Example 5

$$\int_1^3 x^2\,dx = \frac{x^3}{3}\Big|_1^3 = \frac{3^3}{3} - \frac{1^3}{3} = \frac{26}{3} = 8.\overline{6}$$

$\int_1^3 (X^2)dX$

8.666666667

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} -\csc^2 x\,dx = \cot x \Big|_{\pi/4}^{\pi/2}$$

$$= \cot\frac{\pi}{2} - \cot\frac{\pi}{4} = 0 - 1 = -1$$

$\int_{\pi/4}^{\pi/2}(-1/(\sin(X))$▸

-1

$$\int_{-1}^{3} (x^3 + 1)\,dx$$

$\int_{-1}^{3}(X^3+1)dX$

24

$$= \left[\frac{x^4}{4} + x\right]_{-1}^{3} = \left(\frac{81}{4} + 3\right) - \left(\frac{1}{4} - 1\right) = 24$$

Exercise 5

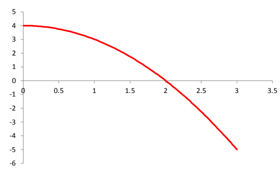
Use part 2 of the Fundamental Theorem of Calculus to evaluate

$$\int_0^1 (x^2 + \sqrt{x})\,dx$$

Then use your graphing calculator to confirm your answer.

Example 6

Find the area of the region between the curve $y = 4 - x^2$, $0 \le x \le 3$, and the x-axis.



The graph crosses the x-axis at $x = 2$. Therefore, partition the function into two regions and integrate each.

$$\int_0^2 (4 - x^2)\,dx = \left[4x - \frac{1}{3}x^3\right]_0^2 = \frac{16}{3}$$

$$\int_2^3 (4 - x^2)\,dx = \left[4x - \frac{1}{3}x^3\right]_2^3 = -\frac{7}{3}$$

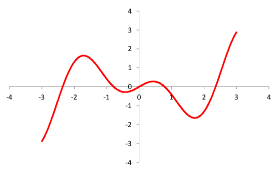
$$Area = \left|\frac{16}{3}\right| + \left|-\frac{7}{3}\right| = \frac{23}{3}$$

Exercise 6

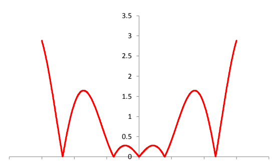
Find the area of the region between the curve $y = 2 - x$, $0 \le x \le 3$, and the x-axis.

Example 7

Find the area of the region between the curve $y = x\cos(2x)$, $-3 \le x \le 3$, and the x-axis.



Find the absolute value of the function and integrate it using a graphing calculator.



$\int_{-3}^{3}(|X\cos(2X)|)dX$

5.425029484

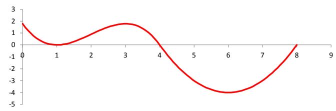
Exercise 7

Find the area of the semielliptical region between the x-axis and the graph of $y = \sqrt{8 - 2x^2}$.

Example 8

The graph of a continuous function f with domain [0, 8] is shown in the figure below. Let h be the function defined by

$$h(x) = \int_1^x f(t)\,dt.$$



1. Find $h(1)$.

   $$h(1) = \int_1^1 f(t)\,dt = 0$$

2. Is $h(0)$ positive or negative?

   $$h(0) = \int_1^0 f(t)\,dt.$$

   Integrating from right to left, and function is positive.

3. Find the value of x for which $h(x)$ is a maximum.

   The derivative of h is positive on $(0, 1)$, positive on $(1, 4)$, and negative on $(4, 8)$. The function is increasing on $[0, 4]$ and decreasing on $[4, 8]$. Therefore, $f(4)$ is a maximum.

4. Find the value of x for which $h(x)$ is a minimum.

   The minimum must occur at an endpoint.

$$h(0) = \int_1^0 f(t)\,dt \approx -0.5$$

$$h(8) = \int_1^8 f(t)\,dt \approx -\frac{1}{2}4 \cdot 4 < -0.5$$

$h(8)$ is the minimum

5. Find the points of inflection on the graph of $y = h(x)$.

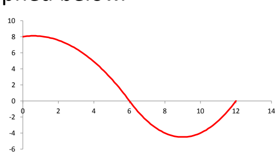
   The points of inflection occur where h'' changes sign. That is where $h' = f$ changes direction: $x = 1, 3, 6$.

Exercise 8

Let

$$H(x) = \int_0^x f(t)\,dt$$

where f is the continuous function with domain $[0, 12]$ graphed below.



1. Find $H(0)$.
2. On what interval is H increasing? Explain.
3. On what interval is the graph of H concave up? Explain.
4. Is $H(12)$ positive or negative? Explain.
5. Where does H achieve its maximum value? Explain.
6. Where does H achieve its minimum value? Explain.

Homework

P 302: 3-51 X 3, 58, 64, 74