Chapter 5.3

Definite Integrals and Antiderivatives

Objective

- Apply the rules for definite integrals and find the average value of a function over a closed interval.

Learning Target

80% of the students will be able to find the area under the function $y = \sin x$ over the interval $[0, \pi]$.

Standard

F-IF.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

Overview

- Properties of Definite Integrals
- Average Value of a Function
- Mean Value Theorem for Definite Integrals
- Connecting Differential and Integral Calculus

Rules for Definite Integrals

1. Order of Integration

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

2. Zero

$$\int_a^a f(x)\,dx = 0$$

3. Constant Multiple

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$

4. Sum and Difference

$$\int_a^b \big(f(x) \pm g(x)\big)\,dx$$
$$= \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

5. Additivity

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

6. Min-Max Inequality

If $\max f$ and $\min f$ are the maximum and minimum of f on the interval $[a, b]$, then

$$\min f \cdot (b-a) \le \int_a^b f(x)\,dx \le \max f \cdot (b-a)$$

7. Domination

$f(x) \ge g(x)$ on $[a, b] \Rightarrow$

$$\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$$

Example 1

Let

$$\int_{-1}^{1} f(x)\,dx = 5$$

$$\int_{1}^{4} f(x)\,dx = -2$$

$$\int_{-1}^{1} h(x)\,dx = 7$$

Find

$$\int_4^1 f(x)\,dx$$

$$= -\int_1^4 f(x)\,dx = -(-2) = 2$$

$$\int_{-1}^4 f(x)\,dx$$

$$= \int_{-1}^1 f(x)\,dx + \int_1^4 f(x)\,dx = 5 + (-2) = 3$$

Find

$$\int_{-1}^1 \big(2f(x) + 3h(x)\big)\,dx$$

$$= 2\int_{-1}^1 f(x)\,dx + 3\int_{-1}^1 \big(h(x)\big)\,dx$$

$$= 2 \cdot 5 + 3 \cdot 7 = 31$$

$$\int_0^1 f(x)\,dx$$

Not enough information

Find

$$\int_{-2}^2 h(x)\,dx$$

Not enough information

$$\int_{-1}^4 \big(f(x) + h(x)\big)\,dx$$

Not enough information

Exercise 1

Let

$$\int_1^9 f(x)\,dx = -1$$

$$\int_7^9 f(x)\,dx = 5$$

$$\int_7^9 h(x)\,dx = 4$$

Find

$$\int_1^9 -2f(x)\,dx$$

$$\int_1^7 f(x)\,dx$$

$$\int_9^7 \big(h(x) - f(x)\big)\,dx$$

Example 2

Show

$$\int_0^1 \sqrt{1+\cos x}\,dx < \frac{3}{2}$$

$$\int_0^1 \sqrt{1+\cos x}\,dx < \max\sqrt{1+\cos x} \cdot (1-0)$$

$$\max\sqrt{1+\cos x} = \sqrt{2} \approx 1.414 < 1.5$$

Exercise 2

Show

$$\int_0^1 \sin(x^2)\,dx \neq 2$$

Mean Value of a Set of Numbers

Suppose we have a set of n numbers,

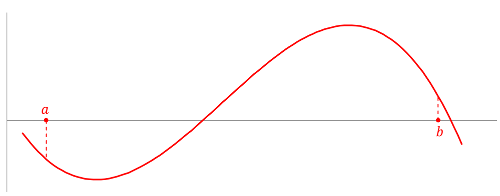
$$\{x_1, x_2, \cdots, x_{n-1}, x_n\}$$

The mean of these n numbers is,

$$\frac{1}{n}\sum_{i=1}^{n} x_i$$

Mean Value of a Function

Suppose we have a function, f, defined on a closed interval $[a, b]$.



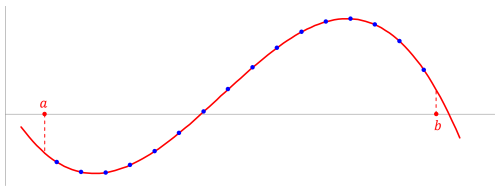
Define a set of n numbers evenly distributed between a and b, $\{x_1, x_2, \cdots, x_{n-1}, x_n\}$.

$$\Delta x = \frac{b-a}{n}$$

$$x_i = a + \Delta x \cdot \left(i - \frac{1}{2}\right)$$

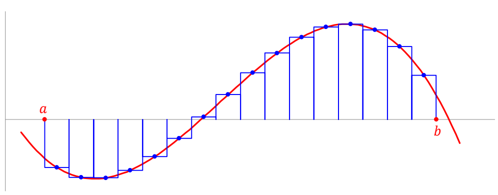
The value of the function at these points is,

$$\{f(x_1), f(x_2), \cdots, f(x_{n-1}), f(x_n)\}$$



The average of the function at these numbers is,

$$\frac{1}{n}\sum_{i=1}^{n} f(x_i) = \frac{\Delta x}{b-a}\sum_{i=1}^{n} f(x_i) = \frac{1}{b-a}\sum_{i=1}^{n} f(x_i)\Delta x$$



Mean Value of a Function

This is a Riemann sum.

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x = \int_a^b f(x)\,dx$$

The average of a function over the closed interval $[a, b]$ is

$$av(f) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Example 3

Find the mean value of $f(x) = 4 - x^2$ on $[0, 3]$.

$$av(f) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$= \frac{1}{3-0}\int_0^3 (4 - x^2)\,dx \qquad \int_0^3 (4-x^2)dx$$

3

$$= \frac{1}{3-0} \cdot 3 = 1$$

Example 3

Does $f(x) = 1$ at some point in the interval $[0, 3]$?

$$f(x) = 4 - x^2 = 1$$

$$x^2 = 3$$

$$x = \sqrt{3}$$

Mean Value Theorem for Definite Integrals

If f is continuous on $[a, b]$ then $\exists c: a \le c \le b$

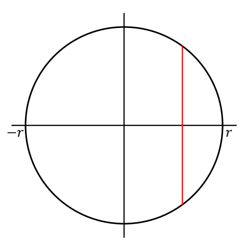
$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Exercise 3

Use a graphing calculator to find the average value of $y = -\frac{x^2}{2}$ on $[0, 3]$.

Average Length of the Chord of a Circle

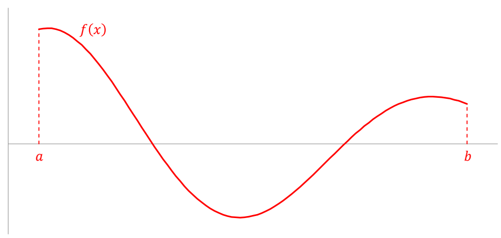
1. Show that the length of the chord at x is $2\sqrt{r^2 - x^2}$
2. Set up an integral expression for the average value of $2\sqrt{r^2 - x^2}$ over $[-r, r]$



3. Evaluate the integral by identifying the integral as an area.
4. What is the average length of a chord of a circle of radius r?
5. Find the value of x when the function assumes this value.
6. Explain how we can use the Mean Value Theorem for Definite Integrals to show that the function assumes this value.

Finding the Derivative of an Integral

In the following steps, use this graph.



1. Choose any x greater than a in the interval $[a, b]$ and mark it on the x-axis.
2. Using only vertical line segments, shade in the region between f and the x-axis from a to x.
3. Your shaded region represents a definite integral

$$\int_a^x f(t)\,dt$$

Why not this integral?

$$\int_a^x f(x)\,dx$$

4. Compare your picture with others produced by members of your group. Notice how your integral depends on which x you chose in the interval $[a, b]$. Therefore, the integral is a function of x on $[a, b]$. Call it $F(x)$.

5. Recall that $F'(x)$ is the limit of $\Delta F/\Delta x$ as Δx gets smaller and smaller. Represent ΔF in your picture by drawing one more vertical shading segment to the right of the one you drew in step 2. ΔF is the area of your vertical segment. (It could be negative.)
6. Represent Δx in your picture by moving x to beneath your newly-drawn segment. That small change in x is the thickness of your vertical segment, Δx.

7. What is now the height of your vertical segment?

   Compare it to $\Delta F/\Delta x$.
8. Can you see why Newton and Leibniz concluded that $F'(x) = f(x)$?

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

If F is an antiderivative of f, then

$$\int_a^x f(t)\,dt = F(x) + C$$

for some constant C.

Setting $x = a$ gives

$$\int_a^a f(t)\,dt = F(a) + C = 0$$

$$C = -F(a)$$

$$\int_a^x f(t)\,dt = F(x) - F(a)$$

Calculating Definite Integrals

Any definite integral can be calculated by simply finding the difference of the antiderivatives of the function at the end points.

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Where $F(x)$ is the antiderivative of $f(x)$.

Antiderivatives

- Definition:
- $F(x)$ is an antiderivative of $f(x)$ if $F'(x) = f(x)$
- $F(x) = x^2$ is an antiderivative of $f(x) = 2x$, since $F'(x) = 2x$
- $F(x) = x^2 + 5$ is also an antiderivative of $f(x) = 2x$, since $F'(x) = 2x$
- We have a family of antiderivatives for the function $f(x) = 2x$ denoted by $F(x) = x^2 + C$ where C is any real number.

Integration

- The process of finding an antiderivative is called integration.
- $\int 2x\,dx$ is called the integral of $f(x) = 2x$ with respect to x.
- This is also called an indefinite integral, since there are an infinite number of answers.
- $\int 2x\,dx = x^2 + C$
- C is the constant of integration.
- $\frac{d}{dx}(x^2 + C) = 2x$ (can verify the answer)

Basic Integration Rules

- $\int k\,dx = kx + C$
- $\int k \cdot f(x)\,dx = k \int f(x)\,dx$
- $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$
- $\int \sin x\,dx = -\cos x + C$
- $\int \cos x\,dx = \sin x + C$

Basic Integration Rules

- $\int \sec^2 x\,dx = \tan x + C$
- $\int \sec x \tan x\,dx = \sec x + C$
- $\int \csc^2 x\,dx = -\cot x + C$
- $\int \csc x \cot x\,dx = -\csc x + C$
- $\int e^x\,dx = e^x + C$
- $\int \frac{1}{x}\,dx = \ln x + C$

Example 4

Find

$$\int_0^{\pi} \sin x\,dx$$

using

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$$F'(x) = \sin x$$

$\sin x$ is the rate of change of $F(x)$.

$$F(x) = -\cos x$$

$$\int_0^{\pi} \sin x \, dx = -\cos \pi - (-\cos 0)$$

$$= -(-1) - (-1)$$

$$= 2$$

Exercise 4

Interpret e^x as a rate of change, and evaluate the following integral:

$$\int_0^1 e^x \, dx$$

Homework

P 290: 1, 3, 4, 6, 8, 10, 11, 14, 15, 18, 19, 22, 23, 28, 29, 34, 36, 40, 43, 51