Chapter 5.2

Definite Integrals

Objective

- Express the area under a curve as a definite integral and as a limit of Riemann sums.
- Compute the area under a curve using a numerical integration procedure.

Learning Target

80% of the students will be able to find the area under the function $y = \sin x$ over the interval $[0, \pi]$.

Standard

F-IF.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

Overview

- Riemann Sums
- Terminology and Notation of Integration
- Definite Integral and Area
- Constant Functions
- Integrals on a calculator
- Discontinuous Integrable Functions

Sigma Notation

$$\sum_{i=1}^{n} a_i$$

Highest number — n

Greek letter sigma stands for "sum"

Term — a_i

Index — i

Starting number — 1

"the sum of a_i from $i = 1$ to n"

Basic Formulas

$$\sum_{i=1}^{n} c = c \cdot n$$

$$\sum_{i=1}^{3} 3 = 3 \cdot 3 = 9$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{3} i = \frac{3(3+1)}{2} = 6$$

Basic Formulas, continued

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{3} i^2 = \frac{3(3+1)(2 \cdot 3+1)}{6} = 14$$

For more information on sigma notation, see my web site:
http://www.mcclenahan.info/sfhs/Transition/Lecture_Notes/1-4_Sigma_Notation.pdf

Example 1

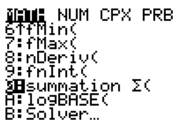
$$\sum_{i=1}^{4} (2 - 5i + 3i^2)$$

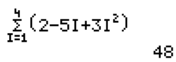
$$= \sum_{i=1}^{4} 2 - 5\sum_{i=1}^{4} i + 3\sum_{i=1}^{4} i^2$$

$$= 2 \cdot 4 - 5 \cdot \frac{4(4+1)}{2} + 3 \cdot \frac{4(4+1)(2 \cdot 4+1)}{6}$$

$$= 8 - 50 + 90 = 48$$

Using a Graphing Calculator





Exercise 1

Evaluate the following sum:

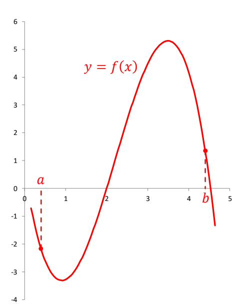
$$\sum_{i=1}^{6}(5+3i-2i^2)$$

Check your answer using a graphing calculator.

Riemann Sums

A sum of rectangles used to approximate the area between the graph of a function and the x-axis.

Consider an arbitrary function defined on a closed interval $[a, b]$.



Partition the interval into n intervals by choosing $n-1$ points:

$a < x_1 < x_2 < x_3 < \cdots < x_{n-2} < x_{n-1} < b$

Let $x_0 \equiv a$ and $x_n \equiv b$

Then, we have the set

$P = \{x_0, x_1, \cdots, x_{n-1}, x_n\}$

This set is a ***partition*** of $[a, b]$.

Notice that the intervals between successive values do not have to be constant.

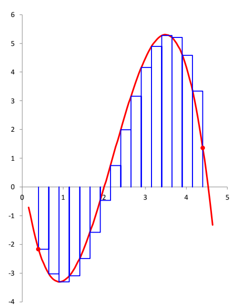
The partition determines n closed ***subintervals***.

These subintervals have widths of

$\Delta x_1, \Delta x_2, \cdots \Delta x_{n-1}, \Delta x_n$

Where

$\Delta x_1 = x_1 - x_0$



Riemann Sum

Now, we perform the sum of all the subintervals:

$$S_n = \sum_{i=1}^{n} f(c_i) \cdot \Delta x_i$$

This is a ***Riemann sum for f on the interval*** $[\boldsymbol{a}, \boldsymbol{b}]$.

As the partitions of become finer, the approximation defined by the rectangles becomes more accurate.

Definite Integral

The width of the each subinterval is Δx_i.

The widest subinterval is called the ***norm***, and is denoted, $\|\Delta\|$.

If a number I exists so that

$$\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i = I$$

then f is ***Integrable*** and I is the ***definite integral*** of f over $[a, b]$

Theorem

- No matter what the choice of the c_i's, the sums always have the same limit so long as the function is continuous.
- ***All continuous functions are Integrable***.
- ***If a function is continuous on an interval*** $[a, b]$***, then its definite integral over*** $[a, b]$ ***exists.***

Definite Integral of a Continuous Function

- Since the choice of c_i's is irrelevant, so long as $\|\Delta\| \to 0$, we can choose a partition, in which all the subintervals have the same width.
- Such a partition is a ***regular partition***.
- The definite integral over $[a, b]$ is,

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x$$

Each c_i is chosen arbitrarily in the i^{th} subinterval.

Notation

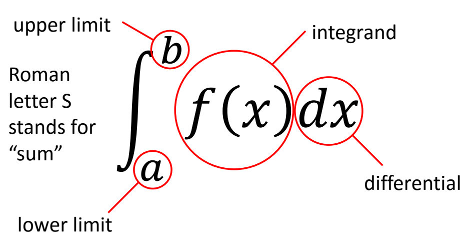
- Leibniz gave us the notation for the derivative:

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{df}{dx}$$

- He also gave us the notation for the integral:

$$\lim_{n \to \infty} \sum_{i=1}^{\infty} f(c_i)\Delta x = \int_a^b f(x)dx$$

Integral Notation



"the integral from a to b of f of x dee x"

Example 2

The interval $[-1, 3]$ is partitioned into n subintervals of equal length $\Delta x = 4/n$.

m_i denotes the midpoint of the i^{th} subinterval.

Express the following limit as an integral:

$$\lim_{n \to \infty} \sum_{i=1}^{n} (3(m_i)^2 - 2m_i + 5)\Delta x$$

$$\int_{-1}^{3} (3x^2 - 2x + 5)dx$$

Exercise 2

The interval $[0, 1]$ is partitioned into n subintervals of equal length $\Delta x = 1/n$.

m_i denotes the midpoint of the i^{th} subinterval.

Express the following limit as an integral:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{4-(m_i)^2}\,\Delta x$$

Area Under a Curve

If $y = f(x)$ is nonnegative and Integrable over a closed interval $[a, b]$, then the area under the curve from a to b is the integral of f from a to b.

$$A = \int_a^b f(x)dx$$

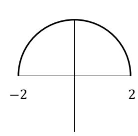
Example 3

We know the area of a semicircle of radius 2 is $A = \frac{1}{2}\pi 2^2 = 2\pi$.

The equation for this semicircle is, $y = \sqrt{4-x^2}$.

Therefore,

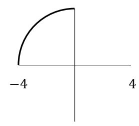
$$\int_{-2}^{2} \sqrt{4-x^2}dx = 2\pi$$



Exercise 3

Use a graph of the integrand and the area to evaluate

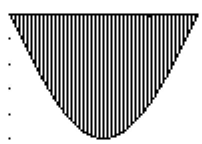
$$\int_{-4}^{0} \sqrt{16 - x^2}dx$$



Area Under a Curve

If $y = f(x)$ is **negative** and Integrable over a closed interval $[a, b]$, then the area under the curve from a to b is the negative of integral of f from a to b.

$$A = -\int_{a}^{b} f(x)dx$$



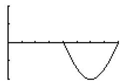
Definite Integral

If $y = f(x)$ is an Integrable function on the interval $[a, b]$, then

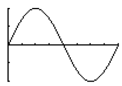
$$\int_{a}^{b} f(x)dx$$
$$= (area\ above\ x - axis)$$
$$- (area\ below\ x - axis)$$

Definite Integrals

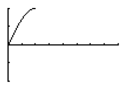
$$\int_{\pi}^{2\pi} \sin x \, dx$$



$$\int_{0}^{2\pi} \sin x \, dx$$

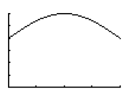


$$\int_{\pi}^{\pi/2} \sin x \, dx$$

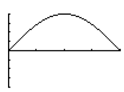


Definite Integrals

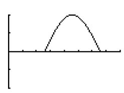
$$\int_{0}^{\pi} (2 + \sin x) dx$$



$$\int_{0}^{\pi} 2 \sin x \, dx$$

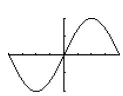


$$\int_{2}^{\pi+2} \sin(x-2) \, dx$$

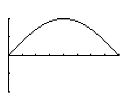


Definite Integrals

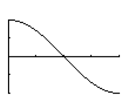
$$\int_{-\pi}^{\pi} \sin u \, du$$



$$\int_{0}^{2\pi} \sin x \, dx$$



$$\int_{0}^{\pi} \cos x \, dx$$



Conjecture

Suppose k is any positive real number.
What is the value of

$$\int_{-k}^{k} \sin x \, dx$$

Support your answer.

Integral of a Constant Function

A constant function, $f(x) = c$, is continuous.
It is Integrable over the interval $[a, b]$.
Use a Riemann sum.

$$\sum_{i=1}^{n} f(c_i) \cdot \Delta x = \sum_{i=1}^{n} c \cdot \frac{b-a}{n}$$

$$= c \cdot (b-a) \sum_{i=1}^{n} \frac{1}{n}$$

$$\sum_{i=1}^{n} f(c_i) \cdot \Delta x = c(b-a) \cdot n \left(\frac{1}{n}\right)$$

$$= c(b-a)$$

This is true for any value of n.

Theorem: The Integral of a Constant

If $f(x) = c$, where c is a constant, on the interval $[a, b]$, then

$$\int_{a}^{b} c \, dx = c(b-a)$$

Example 4

I enter I-25 at 6:30 AM and drive at a steady speed of 80 miles per hour. I exit at 7:00 AM. Express the total distance I have driven as an integral. Then solve the integral,

$$Distance\ Traveled = \int_{6.5}^{7.0} 80\, dt$$

$$= 80 \cdot (7.0 - 6.5) = 40$$

Since the speed is in miles per hour, and the time is in hours, the distance is in miles.

Exercise 4

Find the distance traveled by a car moving at 75 miles per hour from 6:30 AM to 7:02 AM.

Integrating with a Graphing Calculator

Evaluate the following integrals using a graphing calculator.

$$\int_{-1}^{2} x \sin x\, dx$$

$\int_{-1}^{2}(X\sin(X))dX$
2.042759779

$$\int_{0}^{1} \frac{4}{1+x^2}\, dx$$

$\int_{0}^{1}(4/(1+X^2))dX$
3.141592654

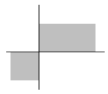
$$\int_{0}^{5} e^{-x^2}\, dx$$

$\int_{0}^{5}(e^{-X^2})dX$
.8862269255

Example 5

Some discontinuous functions are Integrable.

$$\int_{-1}^{2} \frac{|x|}{x}\,dx$$



Using the idea of net area,

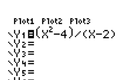
$$\int_{-1}^{2} \frac{|x|}{x}\,dx = -1 + 2 = 1$$

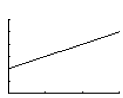
Exercise 5

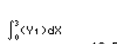
Find the point of discontinuity and evaluate the integral.

$$\int_{-3}^{2} \frac{x}{|x|}\,dx$$

Integrating Discontinuous Functions







Integrating Discontinuous Functions

Explain why

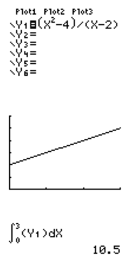
$$f(x) = \frac{x^2 - 4}{x - 2}$$

Is discontinuous.

Use areas to find the integral

$$\int_0^3 \frac{x^2 - 4}{x - 2} dx$$

Use a graphing calculator to confirm.

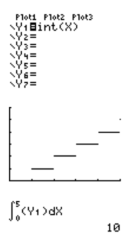


Integrating Discontinuous Functions

Use areas to find the integral

$$\int_0^5 \text{int}(x)\, dx$$

Use a graphing calculator to confirm.



Homework

P 282: 3-27 X 3, 33-35, 37, 40