Chapter 5.1

Estimating with Finite Sums

Objective

- Approximate the area under the graph of a nonnegative continuous function by using rectangle approximation methods.
- Interpret the area under a graph as a net accumulation of a rate of change.

Learning Target

80% of the students will be able to approximate the volume of an ellipsoid using the RAM method.

Standard

F-IF.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

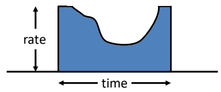
Overview

- Distance Traveled
- Rectangular Approximation Method (RAM)
- Volume of a Sphere
- Cardiac Output

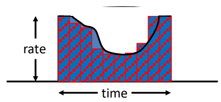
Distance Traveled

- When I leave school, I enter I-25 from St. Francis Dr. and travel for 32 minutes at a constant speed of 1.25 miles per minute until I exit at Bernalillo.
- How far have I driven?
- That's easy: $rate \times time$.
- The area of this rectangle:

- What if I have to slow down for major road construction?



- The distance is still the area of the shape, but how do I calculate it?
- I can subdivide it into rectangles to approximate the area.

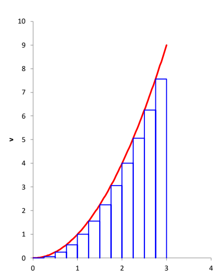


Approximating Distance

- If the rectangles are narrow enough, the sum of their areas is a good approximation of the area under the curve.
- The distance traveled can be approximated by the sum of the area of many small rectangles.

Example 1

- A particle starts at $x = 0$ and moves along the x-axis with velocity $v = t^2$ for time $t \geq 0$. where is the particle at $t = 3$?
- Graph v
- Partition the time interval [0, 3] into subintervals.



Approximating the Area

The area of the i^{th} rectangle is,

$A_i = h_i \Delta t,$

where Δt is the width if the rectangle and

$h_i = \left((i-1)\Delta t\right)^2.$

If the interval is subdivided into n subintervals, then

$\Delta t = \frac{t_{max}}{n} = \frac{3}{n}.$

Approximating the Area

In the sketch above, the total area of the rectangles is,

$$A = \Delta t \sum_{i=1}^{n} \left((i-1)\Delta t\right)^2$$

This can be evaluated without much difficulty.

Rectangular Approximation Method

- Subdividing an area into rectangles and summing the area is called the Rectangular Approximation Method or RAM for short.
- The procedure illustrated above is called the LRAM for Left Rectangular Approximation Method.
- As you can see, this approximation (LRAM) underestimates the area under the curve.

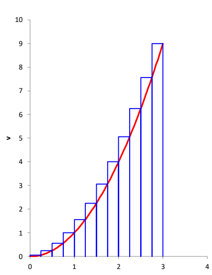
RAM

Rectangular Approximation Method

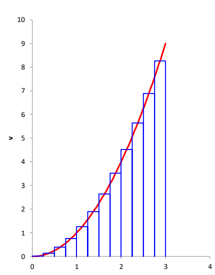
1. Left Rectangular Approximation Method (LRAM)
2. Right Rectangular Approximation Method (RRAM)
3. Midpoint Rectangular Approximation Method (MRAM)

If the width of the rectangles is small enough, the results of all three methods become nearly identical.

RRAM



MRAM



Exercise 1

Use the midpoint approximation to estimate the distance the particle has traveled when $t = 3$.

The width of each rectangle is,

$\Delta t = 0.25$

The midpoints are,

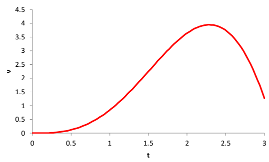
$t = 0.125, 0.375, 0.625, 0.875, 1.125, \ldots$

The height of each rectangle is,

$h = t^2$

Example 2

Let $f(x) = x^2 \sin x$ on the interval [0, 3].



Use the program RIEMANN to compare LRAM, RRAM, and MRAM.

Which Method to Use

It would be a mistake to conclude that RRAM is always smaller than LRAM and that MRAM is better, because it is in between.

This depends on the function and the ease of calculating the values.

Exercise 2

Given the function $y = 2x - x^2$ on the closed interval [0, 2], use a calculator RAM program to complete the following table:

n	LRAM	MRAM	RRAM
10			
50			
100			
500			

RAM Approximation

- A RAM approximation can be used for any nonnegative function.
- It could represent something other than area.
- It could represent, for instance, volume.

Example 3

Estimate the volume of a sphere of radius 4.
Picture the sphere as a series of n disks.
The radius of each disk is $r = \sqrt{16 - x^2}$.
The thickness of each disk is $\Delta x = 8/n$.
The midpoint value of x_i for the i^{th} disk is
$x_i = -4 + 8i/n - 4/n$.
The midpoint radius for the i^{th} disk is
$r_i = \sqrt{16 - {x_i}^2}$.

Now we can approximate the volume of the sphere by using the REIMANN program with the function $f(x) = \pi {r_i}^2 \Delta x = \pi\left(\sqrt{16 - {x_i}^2}\right)^2 \Delta x$

We use the RAM function $\pi(16 - x^2)$ over the interval $[-4, 4]$.

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FUNCTION:
π(16-X²)
LOWER BOUND:-4
UPPER BOUND:4
PARTITIONS:10
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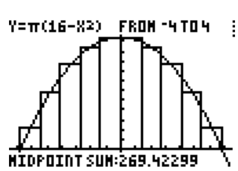


Error

We know that the volume of a sphere is $V = \frac{4}{3}\pi r^3$.

$$\frac{|MRAM_{10} - 256\pi/3|}{256\pi/3} \approx \frac{269.42299 - 268.08257}{268.08257} \approx 0.00500$$

This is one half percent.

Exercise 3

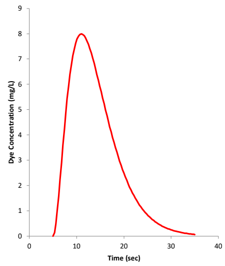
Use MRAM to approximate the volume of a sphere of radius 5. Use $n = 20$.

Example 4

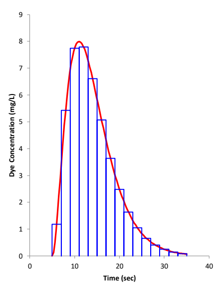
- The volume of blood pumped by a heart in a fixed time interval is called its ***cardiac output***.
- At rest, a human heart will typically pump 5 or 6 liters per minute.
- During strenuous exercise, it might reach as much as 30 liters per minute.
- Disease might alter a persons cardiac output.

Cardiac Output

In order to measure cardiac output, often a dye is injected into a vein near the heart, and the dye concentration is measured in the aorta every few seconds after the dye injection. This graph is a smoothed curve fitted to such data for a healthy heart.



We can estimate the total volume of blood pumped per second by dividing the amount of dye injected by the area under the curve. Since we do not know the function of the curve, we cannot use analytical techniques. Nevertheless, we can draw rectangles and estimate their heights.



Cardiac Output

- We use MRAM and these rectangles to estimate the area under this curve.
- We find $area \approx 55.2\ (^{mg}/_{L})s$.
- Since we injected 5.6 mg of dye, we divide and convert seconds to minutes.
- We estimate the cardiac output to be $6.1\ L/min$.

Exercise 4

Work Exercise 15 on p 270 of the book.

Homework

P 270: 1-8, 10, 13-15, 18, 22, 24, 25, 28, 30