Chapter 4.6

Related Rates

Objective

- Solve related rate problems.

Learning Target

80% of the students will be able to find how fast the top of a 10 ft ladder leaning against a wall is falling if the bottom is being pulled out at a rate of 2 ft/s.

Standard

F-IF.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

Overview

- Related Rate Equations
- Solution Strategy
- Simulating Related Motion

Related Rates

- Sometimes one rate of change is related to another rate of change. All of the problems in this section involve rates of change with respect to time.

Solving a Related Rates Problem

1. Identify the given quantities and rates, and the rate to be determined.
2. Draw a sketch if possible and label all of the quantities and rates.
3. Write an equation relating the quantities in the problem. Any constant here must be constant throughout the problem.

4. Differentiate both sides of the equation with respect to time using implicit differentiation (every variable is a function of time).
5. Substitute in the given quantities and rates, and solve for the unknown rate.
6. Interpret the Solution Translate your mathematical result into the problem setting (with appropriate units) and decide whether the result makes sense.

Example 1

When a balloon is inflated, the volume of the balloon increases with respect to time, and the radius of the balloon also increases with respect to time. Suppose that the radius, r, is increasing at a rate of 2 inches per minute. Find the rate of change of the volume when $r = 6$ inches.

- Given quantities and rates:

  $r = 6 \text{ in}, \quad \frac{dr}{dt} = 2 \, {}^{\text{in}}/_{\text{min}}$
- Equation: $V = \frac{4}{3}\pi r^3$ (volume of a sphere).

  Find $\frac{dV}{dt}$.
- $\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)$
- $\frac{dV}{dt} = \frac{4}{3}\pi\left(3r^2 \cdot \frac{dr}{dt}\right)$

  Differentiate with respect to time

- $\frac{dV}{dt} = \frac{4}{3}\pi\left(3(6 \text{ in})^2 \cdot (2 \, {}^{\text{in}}/_{\text{min}})\right) = 288\pi \, {}^{\text{in}^3}/_{\text{min}}$
- When the balloon's diameter is six inches. Its radius is increasing at a rate of two inches per minute. The balloon's volume is increasing at a rate of 288π cubic inches per minute.

Exercise 1

All edges of a cube are expanding at a rate of 6 centimeters per second. How fast is the surface area changing when each edge is

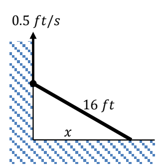
a. 2 cm

b. 10 cm

Example 2

A construction worker pulls a 16-foot plank up the side of a building by means of a rope tied to the end of the plank. Assume the opposite end of the plank follows a path perpendicular to the wall of the building and the worker pulls the rope at a rate of 0.5 $^{\text{ft}}/_{\text{sec}}$. How fast is the end of the plank sliding along the ground when it is 8 feet from the wall of the building?

1. Identify
   a. The ladder is 16 feet long.
   b. It is being pulled up at 0.5 ft/s.
   c. How fast is the base moving when it is 8 feet from the wall?
2. Draw a sketch.



3. Write an equation
   a. $x = 8$ ft,.
   b. $\frac{dy}{dt} = 0.5\ ^{\text{ft}}/_{\text{sec}}$
   c. Find $\frac{dx}{dt}$.
   d. Use the Pythagorean Theorem: $x^2 + y^2 = 16^2$
   e. $y = \sqrt{16^2 - 8^2} = 8\sqrt{3}$

4. Differentiate with respect to time
   a. $\frac{d}{dt}(x^2 + y^2) = \frac{d}{dt}(16^2)$
   b. $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$
   c. $\frac{dx}{dt} = -\frac{2y}{2x} \cdot \frac{dy}{dt} = -\frac{y}{x} \cdot \frac{dy}{dt}.$
5. Substitute
   a. $\frac{dx}{dt} = -\frac{8\sqrt{3}\text{ ft}}{8\text{ ft}} \cdot (0.5\,{}^{\text{ft}}/_{\text{sec}}) = -\frac{\sqrt{3}}{2}\,{}^{\text{ft}}/_{\text{sec}}.$

6. Interpret the Solution

   When the bottom of the ladder is 8 feet from the wall, it is moving toward the wall at $\sqrt{3}/2\ \ ft/s.$

Exercise 2

A ladder 25 feet long is leaning against the wall of a building. The base of the ladder is pulled away from the wall at a rate of 2 ft/s.

How fast is the top of the ladder moving down when its base is

a. 7 feet from the wall?
b. 15 feet from the wall?
c. 24 feet from the wall?

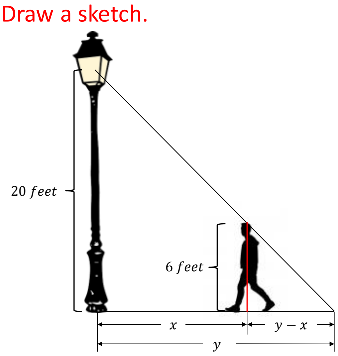
Example 3

A man 6 feet tall is walking at 5 feet per second directly toward a lamp post 20 feet tall.

a. When he is 10 feet from the base of the light, at what rate is the tip of his shadow moving?
b. When he is 10 feet from the base of the light, at what rate is the length of his shadow changing?

1. Identify
   a. The light is 20 feet high.
   b. The man is walking toward the light at $5\ ft/s$.
   c. How fast is the tip of the man's shadow moving when he is 10 feet from the light?
   d. How fast is the length of the man's shadow changing when he is 10 feet from the light?

2. Draw a sketch.



3. Write an equation

Similar triangles: the ratios of corresponding parts are equal.

a. $\frac{20}{6} = \frac{y}{y-x}$

b. $20(y-x) = 6y$

c. $20y - 20x = 6y \Rightarrow y = \frac{10}{7}x$

4. Differentiate with respect to time

a. $\frac{dy}{dt} = \frac{10}{7} \cdot \frac{dx}{dt}$

b. $\frac{d}{dt}(y-x) = \frac{dy}{dt} - \frac{dx}{dt}$

5. Substitute

a. $\frac{dy}{dt} = \frac{10}{7}(-5\ {}^{\text{ft}}/_{\text{sec}}) = -\frac{50}{7}\ {}^{\text{ft}}/_{\text{sec}}$

b. $\frac{d}{dt}(y-x) = -\frac{50}{7} + 5 = -\frac{15}{7}\ {}^{\text{ft}}/_{\text{sec}}$

6. Interpret the Solution

When a 6 foot man is 10 feet from a 20 foot high light pole and is moving toward it at $5\ ft/s$, the tip of his shadow is moving toward the pole at $\frac{50}{7}\ ft/s$.

The length of his shadow is decreasing at a rate of $\frac{15}{7}\ ft/s$.

Exercise 3

A man 6 feet tall is walking at 5 feet per second away from a lamp post 15 feet tall.

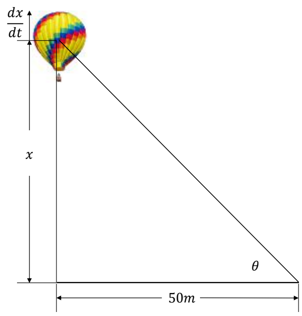
a. When he is 10 feet from the base of the light, at what rate is the tip of his shadow moving?
b. When he is 10 feet from the base of the light, at what rate is the length of his shadow changing?

Example 4

A balloon rises at a rate of $4\ ^{\mathrm{m}}/_{\mathrm{sec}}$ from a point on the ground 50 meters from an observer. Find the rate of change of the angle of elevation of the balloon from the observer when the balloon is 50 meters above the ground.

1. Identify
    a. The balloon is 50 meters from an observer.
    b. It rises at a rate of $4\ m/s$.
    c. What is the rate of change of the angle of elevation of the balloon from the observer when the balloon is 50 meters above the ground?

2. Draw a sketch.



3. Write an equation

a. θ is the angle of elevation

b. $\tan\theta = \frac{x}{50}$

c. $\frac{dx}{dt} = 4\ m/s$

d. $x = 50\ m$

4. Differentiate with respect to time

a. $\frac{d}{dt}(\tan\theta) = \frac{d}{dt}\left(\frac{x}{50}\right)$

b. $\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{50} \cdot \frac{dx}{dt}$

c. $\frac{d\theta}{dt} = \frac{\cos^2\theta}{50} \cdot \frac{dx}{dt}$

5. Substitute

a. $\tan\theta = \frac{50}{50} = 1 \Rightarrow \theta = 45°$

b. $\frac{d\theta}{dt} = \frac{\cos^2(45°)}{50} \cdot 4\ \mathrm{rad}/_{\mathrm{s}}$

c. $\frac{d\theta}{dt} = \frac{\left(\frac{\sqrt{2}}{2}\right)^2}{50} \cdot 4\ \mathrm{rad}/_{\mathrm{s}} = \frac{1}{25}\ \mathrm{rad}/_{\mathrm{s}}$

6. Interpret the Solution

If a balloon rises at a rate of $4\ ^{\mathrm{m}}/_{\mathrm{sec}}$ from a point on the ground 50 meters from an observer, the rate of change of the angle of elevation when the balloon is 50 meters above the ground is $1/25\ rad/s$.

Exercise 4

An airplane flies at an altitude of 5 miles toward a point directly above an observer on the ground. The speed of the airplane is 600 miles per hour. Find the rates at which the angle of elevation θ is changing when the angle is

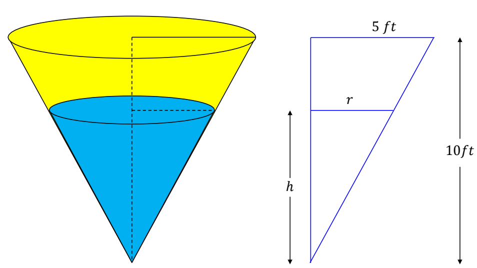
a. $\theta = 30°$
b. $\theta = 60°$
c. $\theta = 75°$

Example 5

Water runs into a conical tank at the rate of $9\ ft^3/min$. The tank stands point down and has a height of $10\ ft$ and a base radius of $5\ ft$. How fast is the water level rising when the water is $6\ ft$ deep?

1. Identify
   a. A conical tank is filling with water.
   b. Let V be the volume of the cone of water, r is its radius, and h is its height.
   c. It fills at a rate of $\frac{dV}{dt} = 9\ ft^3/min$.
   d. What is the rate of change of the height of the cone of *water* inside the tank when the height of that cone is 6 ft?

2. Draw a sketch.



3. <u>Write an equation</u>
   a. The volume of the cone of water is $V = \frac{1}{3}\pi r^2 h$
   b. Using similar triangles, $\frac{r}{h} = \frac{5}{10}$
   c. $r = \frac{1}{2}h$
   d. $V = \frac{1}{3}\pi\left(\frac{1}{2}h\right)^2 h = \frac{\pi}{12}h^3$

4. Differentiate with respect to time
   a. $\frac{dV}{dt} = \frac{\pi}{12} \cdot 3h^2 \frac{dh}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt}$
   b. $\frac{dh}{dt} = \frac{4}{\pi h^2} \frac{dV}{dt}$

5. Substitute
   a. $\frac{dh}{dt} = \frac{4}{\pi 6^2} \cdot 9$
   b. $\frac{dh}{dt} = \frac{1}{\pi} \approx 0.32$

6. Interpret the Solution

   When the water level in the tank is 6 ft deep, the water level is rising at $0.32\ ft/min$.

Exercise 5

A horizontal trough is 12 feet long and 3 feet across the top. Its ends are isosceles triangles with altitudes of 3 feet.

a. Water is being pumped into the trough at 2 cubic feet per minute. How fast is the water level rising when the depth h is 1 foot?

b. The water is rising at a rate of $\frac{3}{8}$ inch per minute when the depth is 2 feet. Determine the rate at which water is being pumped into the trough.

Homework

P 251: 3-42 multiples of 3, 43

Acknowledgement

I wish to thank David Platt of the Front Range Community College, Fort Collins, CO for graciously allowing me to incorporate portions of his Power Point presentations into my own.