Chapter 4.5

Linearization and Newton's Method

Objective

- Find linearizations and use Newton's method to approximate the zeros of a function.
- Estimate the change in a function using differentials.

Learning Target

80% of the students will be able to use Newton's method to estimate all real solutions of the equation $x^4 + x - 3 = 0$ to an accuracy of 6 decimal places.

Standard

F-IF.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

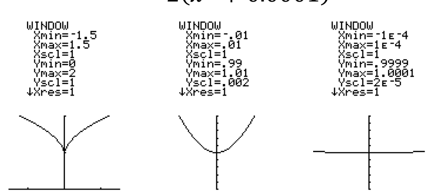
Overview

- Linear Approximations
- Newton's Method
- Differentials
- Estimating Change with Differentials
- Absolute, Relative, and Percentage Change
- Sensitivity to Change

Local Linearity

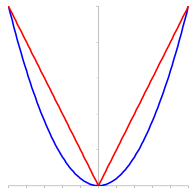
$$f(x) = (x^2 + 0.0001)^{1/4} + 0.9$$

$$f'(x) = \frac{x}{2(x^2 + 0.0001)^{3/4}}$$



Local Linearity

- If we zoom-in far enough at a point on the graph of a function, the graph will appear linear, except at a cusp.
- In general, if a function f is differentiable at a point, then the graph of f is "locally linear" at that point.



Tangent Line Approximation

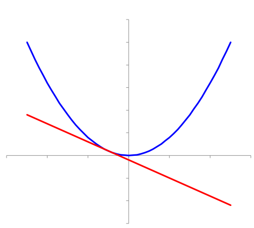
In general, the tangent line to the graph of f at the point $(c, f(c))$ is

$$y = f(c) + f'(c)(x - c)$$

where $f'(c)$ is the slope of the tangent line at $(c, f(c))$.

Tangent Line Approximation

- If we draw a tangent line to the graph of f at a point where f is differentiable, then the tangent line can be used to approximate f for points close to the tangent point.



Linearization

If f is differentiable at $x = a$, then the equation of the tangent line,

$$L(x) = f(a) + f'(a)(x - a),$$

defines the ***linearization*** of f at a. The approximation $f(x) \approx L(x)$ is the ***standard linear approximation*** of f at a. The point $x = a$ is the ***center*** of the approximation.

Example 1

Find the linearization of $f(x) = \sqrt{1 + x}$, and use it to approximate $\sqrt{1.02}$ without a calculator. Then use a calculator to determine the accuracy of the approximation.

1. Since $f(0) = 1$, the point of tangency is $(0, 1)$.
2. Since $f'(x) = \frac{1}{2}(1 + x)^{-1/2}$, the slope of the tangent line is $f'(0) = \frac{1}{2}$.

3. Therefore,

$$L(x) = \frac{1}{2}(x - 0) + 1 = 1 + \frac{x}{2}$$

4. To approximate $\sqrt{1.02}$, we use $x = 0.02$.

$$\sqrt{1.02} = f(0.02) \approx L(0.02) = 1 + \frac{0.02}{2} = 1.01$$

```
√1.02
        1.009950494
Ans-1.01
     -4.95061638E-5
```

Exercise 1

Find the linearization of $f(x) = x^3 - 2x + 3$ at $x = 2$, and use it to approximate

$$2.1^3 - 2 \cdot 2.1 + 3$$

without a calculator. Then use a calculator to determine the accuracy of the approximation.

If I have a calculator, when will I ever need this?

- Many physical models are nonlinear and have multiple parameters.
- Plasma phenomena.
- Cooling phenomena.

Example 2

Find the linearization of $f(x) = \cos x$ at $x = \pi/2$, and use it to approximate $\cos 1.75$ without a calculator. Then use a calculator to determine the accuracy of the approximation.

1. Since $f(\pi/2) = \cos(\pi/2) = 0$, the point of tangency is $(\pi/2\,, 0)$.
2. Since $f'(x) = -\sin x$, the slope of the tangent line is $f'(\pi/2) = -\sin(\pi/2) = -1$.

3. Therefore,
$$L(x) = (-1)\left(x - \frac{\pi}{2}\right) + 0 = -x + \frac{\pi}{2}.$$
4. To approximate $\cos(1.75)$, we use $x = 1.75$.
$$\cos 1.75 = f(1.75) \approx L(1.75) = -1.75 + \frac{\pi}{2}$$

```
cos(1.75)
     -.1782460556
-1.75+π/2-Ans
   -9.57617556E-4
```

Exercise 2

Find the linearization of $f(x) = \ln(x+1)$ at $x = 0$, and use it to approximate
$$\ln(1.01)$$
without a calculator. Then use a calculator to determine the accuracy of the approximation.

Example 3
Approximating Binomial Powers

$$(1+x)^k \approx 1 + kx$$

$$\sqrt[3]{1-x} = \left(1 + (-x)\right)^{1/3} \approx 1 + \frac{1}{3}(-x) = 1 - \frac{1}{3}x$$

$$\frac{1}{1-x} = \left(1 + (-x)\right)^{-1} \approx 1 + (-1)(-x) = 1 + x$$

$$\sqrt{1+5x^4} = \left(1 + (5x^4)\right)^{1/2} \approx 1 + \frac{1}{2}(5x^4) = 1 + \frac{5}{2}x^4$$

$$\frac{1}{\sqrt{1-x^2}} = \left(1 + (-x^2)\right)^{-1/2} \approx 1 + \left(-\frac{1}{2}\right)(-x^2) = 1 + \frac{1}{2}x^2$$

Exercise 3

Use the linear approximation $(1+x)^k \approx 1 + kx$ to find an approximation for:

$$\frac{1}{\sqrt[3]{1+x}}$$

$$(3+2x)^{1/2}$$

Example 4
Approximating Roots

a. $\sqrt{123}$

Let $f(x) = \sqrt{x}$. The closes perfect square to 123 is 121; therefore we center the linearization at 121. The tangent line at (121, 11) has slope

$$f'(121) = \frac{1}{2}(121)^{-1/2} = \frac{1}{2}\cdot\frac{1}{\sqrt{121}} = \frac{1}{22}$$

$$\sqrt{123} \approx L(123) = 11 + \frac{1}{22}(123-121) = 11.\overline{09}$$

Example 4
Approximating Roots

b. $\sqrt[3]{123}$

Let $f(x) = \sqrt[3]{x}$. The closes perfect cube to 123 is 125; therefore we center the linearization at 125. The tangent line at (125, 5) has slope

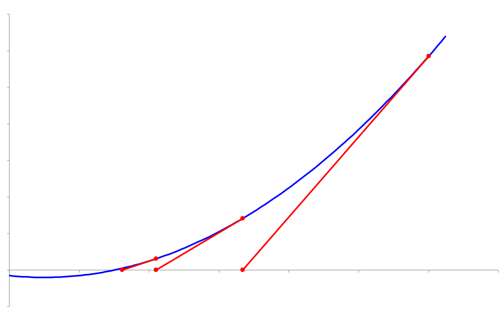
$$f'(125) = \frac{1}{3}(125)^{-2/3} = \frac{1}{3}\cdot\frac{1}{\left(\sqrt[3]{125}\right)^2} = \frac{1}{75}$$

$$\sqrt[3]{123} \approx L(123) = 5 + \frac{1}{75}(123-125) = 4.97\overline{3}$$

Exercise 4

Approximate $\sqrt[3]{26}$ by using a linearization centered at an appropriate nearby number.

Newton's Method



Procedure for Newton's Method

1. Guess a first approximation to a solution of the equation $f(x) = 0$. A graph of $y = f(x)$ may help.
2. Use the first approximation to get a second using,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

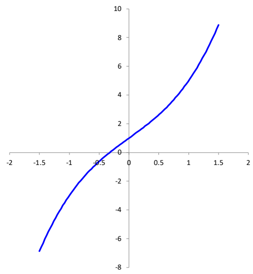
3. Repeat until the desired precision is obtained.

Example 5

Use Newton's method to solve $x^3 + 3x + 1 = 0$.

Let $f(x) = x^3 + 3x + 1$,

Then $f'(x) = 3x^2 + 3$.



$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{{x_n}^3 + x_n + 1}{3{x_n}^2 + 3}$$

The graph suggests that $x_1 = -0.3$ is a good place to start.

```
-.3→X
                  -.3
X-(X³+3X+1)/(3X²▸
          -.322324159
```

```
Ans→X
        -.3221853546
X-(X³+3X+1)/(3X²▸
        -.3221853546
```

```
Ans→X
        -.3221853603
X-(X³+3X+1)/(3X²▸
        -.3221853546
```

```
Ans→X
         -.322324159
X-(X³+3X+1)/(3X²▸
        -.3221853603
```

Let's try an exploration using your graphing calculator.

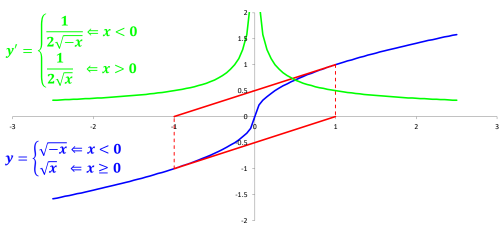
1. Enter $Y_1 = X^3 + 3X + 1$.
2. Enter $Y_2 = 3X^2 + 3$.
3. On the home screen enter $-.3 \rightarrow X$.
4. Enter $X - Y_1/Y_2 \rightarrow X$.
5. Press the ENTER key over an over. Watch what happens.
6. Try different values for the initial guess.

Exercise 5

Use Newton's method to estimate all real solutions to $x^4 + x - 3 = 0$. make your answers accurate to 6 decimal places.

Newton Fails

Sometimes, Newton's method will fail to converge, as shown in this example.



Differentials

$\frac{dy}{dx}$ looks like a quotient of real numbers.

In reality, $\frac{dy}{dx}$ is a quotient of limits.

$\frac{dy}{dx} = f'(x)$; therefore, we define differentials:

The differential dx is an independent variable.

The differential dy is a dependent variable.

$$dy = f'(x)dx$$

- Find the differential dy of the function.
- $y = \sqrt{x^2 - 4} = (x^2 - 4)^{\frac{1}{2}}$
- $dy = \frac{1}{2}(x^2 - 4)^{-\frac{1}{2}}(2x)dx = \frac{x}{\sqrt{x^2-4}}dx$

- $\frac{dy}{dx}$ represents $f'(x)$ (notation for the derivative).
- $\frac{dy}{dx}$ also represents the quotient of two differentials.
- $\therefore \frac{dy}{dx} = f'(x) \Longrightarrow dy = f'(x)dx$

Example 6

Find the differential dy and evaluate it for the given values of x and dx.

$y = x^5 + 37x, \qquad x = 1, \qquad dx = 0.01$

$$dy = y'dx = (5x^4 + 37) \cdot dx$$
$$= (5(1)^4 + 37) \cdot 0.01 = 0.42$$

$y = \sin 3x, \qquad x = \pi, \qquad dx = -0.02$

$$dy = y'dx = (3\cos 3x) \cdot dx$$
$$= (3\cos 3\pi) \cdot (-0.02) = 0.06$$

$x + y = xy, \qquad x = 2, \qquad dx = 0.05$

Use implicit differentiation:

$$\frac{dx}{dx} + \frac{dy}{dx} = \frac{dx}{dx}y + x\frac{dy}{dx}$$
$$\frac{dy}{dx}(1 - x) = y - 1$$
$$dy = \frac{y-1}{1-x}dx$$
$$x = 2 \Rightarrow y = 2$$
$$dy = \frac{2-1}{1-2} \cdot (0.05) = -0.05$$

Exercise 6

Find the differential dy and evaluate it for the given values of x and dx.

$y = x^3 - 3x, \quad x = 1, \quad dx = 0.01$

Differential Rules

- Product Rule –
- $d(uv) = \frac{d}{dx}(uv)\ dx$ use the product rule
- $d(uv) = \left[u\frac{d}{dx}(v) + v\frac{d}{dx}(u)\right] dx$
- $d(uv) = u\left(\frac{d}{dx}(v)dx\right) + v\left(\frac{d}{dx}(u)dx\right)$
- $d(uv) = udv + vdu$
- Quotient Rule: $d\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2}$
- $d(cu) = cdu$ where c is a constant.
- $d(u \pm v) = du \pm dv$

Error Propagation Applications

- Let x be the measured value of a variable.
- Let $x + \Delta x$ represent the exact value
- Δx is the error in measurement
- If the measured value x is used to compute another value $f(x)$, then the error Δx will affect the value of $f(x)$.
- The difference between $f(x + \Delta x)$ and $f(x)$ is called the propagated error, denoted Δy.
- $\Delta y = f(x + \Delta x) - f(x)$

- $\Delta y = f(x + \Delta x) - f(x)$ is approximated by the differential dy, where $dy = f'(c)dx$.
- This is the possible propagated error.
- To determine if dy is large or small, relative to y, find the relative error $\frac{dy}{y}$
- Expressed as a percent, the relative error is called the percentage error $\frac{dy}{y} \cdot 100\%$

Example 7

- The measurements of the base and altitude of a triangle are 36 and 50 cm, respectively. The possible error in each measurement is 0.25 cm. Use differentials to approximate the possible error in computing the area of the triangle.
- $A = \frac{1}{2} b \cdot h, \quad b = 36, \quad h = 50, \quad db = dh = \pm 0.25$
- $dA = \frac{1}{2} b \cdot dh + \frac{1}{2} h \cdot db$ (product rule)

- Possible propagated error:
- $dA = \frac{1}{2}(36)(\pm 0.25) + \frac{1}{2}(50)(\pm 0.25) = \pm 10.75 \text{ cm}^2$
- Relative error: $\frac{dA}{A} = \frac{\pm 10.75}{\frac{1}{2}(36)(50)} = \pm 0.01194$
- Percentage error: $\pm 1.194\ \%$

Exercise 7

The measurement of the circumference of a circle is found to be 64 cm with a possible error of 0.9 cm.

Approximate the percent error in computing the area of the circle.

Example 8

Determining Tolerance

About how accurately should we measure the radius r of a sphere in order to calculate its surface area $S = 4\pi r^2$ within 1% of its actual value?

That is,

$$|\Delta S| \leq \frac{1}{100} S = \frac{4\pi r^2}{100}$$

First, find dS

$$dS = \frac{dS}{dr} dr = 8\pi r \cdot dr$$

Replacing ΔS by dS

$$|8\pi r \cdot dr| \leq \frac{4\pi r^2}{100}$$

$$|dr| \leq \frac{1}{8\pi r} \cdot \frac{4\pi r^2}{100} = \frac{1}{2} \cdot \frac{r}{100} = 0.005r$$

Exercise 8

From exercise 7, Estimate the maximum allowable percent error in measuring the circumference if the error in computing the area of the circle cannot exceed 3%.

Example 9

- Approximate $\sqrt[4]{16.3}$ using differentials
- $f(x) = \sqrt[4]{16.3}, x = 16, dx = 0.3$
- $f(x) = x^{\frac{1}{4}} \Longrightarrow f'(x) = \frac{1}{4}x^{-\frac{3}{4}}$
- $dy = \frac{1}{4}x^{-\frac{3}{4}}dx$
- $dy = \frac{1}{4}\left(16^{-\frac{3}{4}}\right)(0.3) = \frac{1}{32}(0.3) = .009375$
- $f(16 + 0.3) \approx \sqrt[4]{16} + dy = 2.009375$

Exercise 9

Approximate $\sqrt[3]{28}$, $(2.99)^3$

Homework

P 242: 3, 5-10, 11, 14, 15, 18, 19, 22, 25, 28, 31, 34, 37, 40, 47, 52, 64, 65