Chapter 4.4

Modeling and Optimization

Objective

- Solve application problems involving finding minimum or maximum values of functions.

Learning Target

80% of the students will be able to design a one-liter, cylindrical can using the minimum amount of material.

Standard

F-IF.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

Overview

- Examples from Mathematics
- Examples from Business and Industry
- Examples from Economics
- Modeling Discrete Phenomena and Differentiable Functions

Solving Optimization Applications

1. Understand the problem.
   a. Read the problem carefully.
   b. Draw a sketch.
   c. Identify the variables.
2. Form an objective function to be maximized or minimized (primary equation).
3. Form a secondary equation relating the variables and the information given in the problem.

Solving Optimization Applications

4. Solve the secondary equation for one of the variables and substitute the result into the primary equation so that the primary equation has only one independent variable.
5. Graph the primary equation. Determine what values of the function make sense in the problem.
6. Differentiate the primary equation and set equal to zero to find the maximum or minimum value of the dependent variable.

Solving Optimization Applications

7. Solve the mathematical model.
8. Verify that your result is a maximum or minimum using the Second Derivative Test.
9. Express your answer completely.
10. Interpret the solution. Translate your mathematical result into the problem setting and decide whether the result makes sense.

Example 1

Find two nonnegative numbers whose sum is 20 and whose product is as large as possible.

Solution:

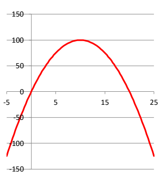
Let one number be x.

The other is $(20 - x)$

$f(x) = x(20 - x)$

Maximum at $x = 10$.

$20 - 10 = 10$

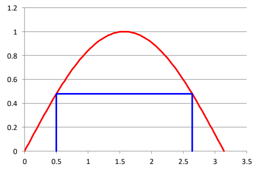


Exercise 1

Find two nonnegative numbers whose sum is 20 and the sum of their squares is as small as possible.

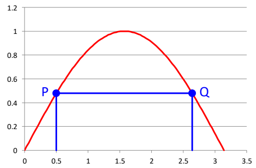
Example 2

A rectangle is inscribed under one arch of a sine curve. What is the largest area the rectangle can have? What are the dimensions of the rectangle?



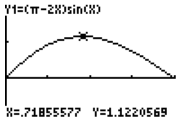
Model

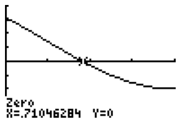
1. Let $(x, \sin x)$ be the coordinates of point P
2. The coordinates of point Q are $(\pi - x, \sin x)$
3. Length $l = \pi - 2x$
4. Area $A = (\pi - 2x)\sin x$



Solve

1. The only critical points occur at the end points and
$A'(x) = -2\sin x$
$+(\pi - 2x)\cos x = 0$
2. Find the zero graphically.
3. Maximum at $x \approx 0.71$.



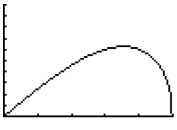


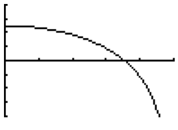
Interpret

1. Area: $A(0.71) \approx 1.12$
2. Width: $w \approx \pi - 2 \cdot 0.71 \approx 1.72$
3. Height: $h \approx \sin 0.71 \approx 0.65$

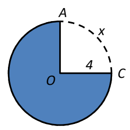
Exercise 2

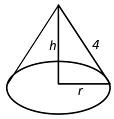
What is the largest possible area of a right triangle whose hypotenuse is 5?





Constructing Cones





1. Show

$$r = \frac{8\pi - x}{2\pi}, \qquad h = \sqrt{16 - r^2}$$

$$V(x) = \frac{\pi}{3}\left(\frac{8\pi - x}{2\pi}\right)^2 \sqrt{16 - \left(\frac{8\pi - x}{2\pi}\right)^2}$$

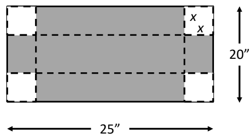
Constructing Cones

2. Show that the natural domain of V is $0 \le x \le 16\pi$. Graph V over this domain.
3. Explain why the restriction $0 \le x \le 8\pi$ makes sense in the problem situation. Graph V over this domain.
4. Use graphical methods to find where the cone has its maximum volume, and what that volume is.
5. Confirm your findings analytically. (Hint: use $V(x) = (1/3)\pi r^2 h$, $h^2 + r^2 = 16$, and the chain rule.

Optimization

- Maximize or minimize something to create the most desirable result.
- Greatest profit
- Least cost

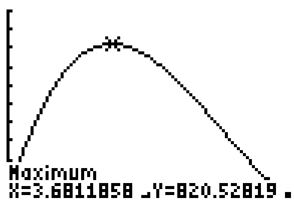
Example 3



- Fold tabs up to make an open top box.
- $V(x) = x(20 - 2x)(25 - 2x)$

Solve Graphical

$$0 \leq x \leq 10$$



Solve Analytically

$$V(x) = 4x^3 - 90x^2 + 500x$$

$$V'(x) = 12x^2 - 180x + 500$$

$$V'(x) = 0$$

$$c = \frac{180 \pm \sqrt{180^2 - 48 \cdot 500}}{24}$$

$$c \approx 3.681, \qquad 11.319$$

Critical point value: $V(3.681) \approx 820.528$

Endpoint values: $V(0) = V(10) = 0$

To get maximum volume, cut out squares with side $\sim 3.68\ in$ from all four corners and fold up.

Exercise 3

Make an open box from a cardboard rectangle by cutting out congruent squares from each corner and folding up the sides.

What are the dimensions of the box with the largest volume you can make?

What is its volume?

Designing a Can

Design a 1 liter can shaped like a right circular cylinder.

What are the dimensions that will use the least material?

Volume (all dimensions in centimeters):

$\pi r^2 h = 1000$

Surface area:

$A = \pi r^2 + 2\pi r h$

Minimize Surface Area

Minimize surface area subject to constraint

$$\pi r^2 h = 1000$$

$$h = \frac{1000}{\pi r^2}$$

$$A = 2\pi r^2 + \frac{2000}{r}$$

Solve Analytically

$$\frac{dA}{dr} = 4\pi r - \frac{2000}{r^2} = 0$$

$$r = \sqrt[3]{\frac{500}{\pi}} \approx 5.419$$

$$\frac{d^2A}{dr^2} = 4\pi + \frac{4000}{r^3} > 0$$

Therefore, $r \approx 5.419$ is a minimum.

$$h = 2r$$

Exercise 4

Design an open top stainless steel tank that has the minimum weight.

1. Square base
2. 500 ft^3 volume
3. Constructed by welding steel plates together

What dimensions should the tank have?

Economics

$r(x) = the\ revenue\ from\ selling\ x\ items,$

$c(x) = the\ cost\ of\ producing\ x\ items,$

$p(x) = r(x) - c(x) =$
$the\ profit\ from\ selling\ x\ items.$

Economics

$\frac{dr}{dx} = marginal\ revenue,$

$\frac{dc}{dx} = marginal\ cost,$

$\frac{dp}{dx} = marginal\ profit.$

Maximum Profit

Maximum profit occurs when marginal revenue equals marginal cost.

$$p' = r' - c' = 0$$

$$r' = c'$$

Could also be maximum loss.

Example 5

$r(x) = 9x,$

$c(x) = x^3 - 6x^2 + 15x,$

x represents thousands of units.

Is there a production level that maximizes profit?

If so, what is it?

$$r' = 9$$

$$c' = 3x^2 - 12x + 15$$

$$3x^2 - 12x + 15 = 9$$

$$x^2 - 4x + 2 = 0$$

$$x = \frac{4 \pm \sqrt{4^2 - 4 \cdot 1 \cdot 2}}{2 \cdot 1}$$

$$x = 2 \pm \sqrt{2} \approx 0.586, \qquad 3.414$$

Maximum or Minimum?

$$p'' = r'' - c''$$

$$p'' = 0 - (6x - 12) = 12 - 6x$$

$$p''(0.586) = 8.484 > 0$$

$$p''(3.414) = -8.484 < 0$$

Maximum profit occurs at $x = 3.414$.

Minimum Average Cost

The average cost is smallest when the average cost equals marginal cost.

Assume that $c(x)$ is differentiable.

$c(x) = cost\ of\ producing\ x\ items, x \geq 0$

$\frac{c(x)}{x} = average\ cost\ of\ producing\ x\ items$

If the average cost can be minimized, then

$$\frac{d}{dx}\left(\frac{c(x)}{x}\right) = 0$$

$$\frac{xc'(x) - c(x)}{x^2} = 0$$

$$xc'(x) - c(x) = 0$$

$$c'(x) = \frac{c(x)}{x}$$

Example 6

Suppose $c(x) = x^3 - 6x^2 + 15x$, where x represents thousands of units.

Is there a production level that minimizes production cost?

If so, what is it?

Look for production levels where marginal cost equals average cost,

Marginal cost:

$$c'(x) = 3x^2 - 12x + 15$$

Average cost:

$$\frac{c(x)}{x} = x^2 - 6x + 15$$

$$3x^2 - 12x + 15 = x^2 - 6x + 15$$

$$2x^2 - 6x = 0$$

$$x = 0, \qquad 3$$

Candidate production level:

$$x = 3$$

Second derivative test:

$$\frac{d}{dx}\left(\frac{c(x)}{x}\right) = 2x - 6$$

$$\frac{d^2}{dx^2}\left(\frac{c(x)}{x}\right) = 2 > 0$$

The second derivative is positive for all x; therefore $x = 3$ gives an absolute minimum.

Exercise 6

Suppose $c(x) = x^3 - 10x^2 - 30x$, where x represents thousands of units.

Is there a production level that minimizes production cost?

If so, what is it?

Rationale

How can we use differentiable functions to represent functions of integers?

If the numbers are large, then the a continuous approximation is close to the actual function.

If the result is not an integer, round to the nearest integer.

Homework

P 226: 1, 5, 9, 12, 17, 19, 20, 27, 31, 36, 40, 41, 43, 45, 46, 48, 50, 57, 60