Chapter 4.3

Connecting f' and f'' with the Graph of f

Objective

- Use the First and Second Derivative Tests to determine the local Extreme Values of a Function.
- Determine the Concavity of a Function and locate the points of inflection by analyzing the second derivative.
- Be able to graph f by using information about f'.

Learning Target

80% of the students will be able to find the local extreme values of the following function:

$$f(x) = x^3 - 12x - 5$$

using the Second Derivative Test.

Standard

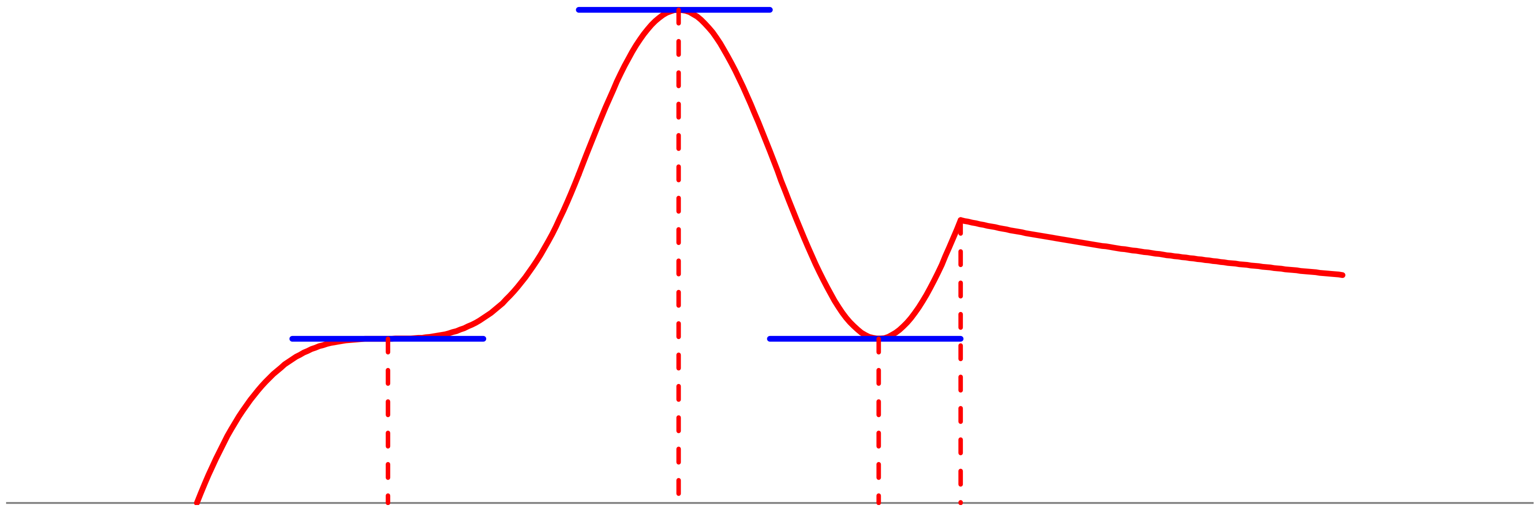
F-IF.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

Overview

- First Derivative Test for Local Extrema
- Concavity
- Points of Inflection
- Second Derivative Test for Local Extrema
- Learning about Functions from Derivatives

First Derivative Test

A function may have local extrema at some critical points, but not have local extrema at others.



First Derivative Test for Local Extrema

- c is a critical number of f, which is continuous on the open interval containing c. If f is differentiable on the interval, except maybe at c,

1. If $f'(x)$ changes from <u>negative to positive</u> at c, then $f(c)$ is a <u>relative minimum</u> of f.
2. If $f'(x)$ changes from <u>positive to negative</u> at c, then $f(c)$ is a <u>relative maximum</u> of f.

Example 1a

$f(x) = x^3 + 3x^2$

$f'(x) = 3x^2 + 6x$

Critical values at

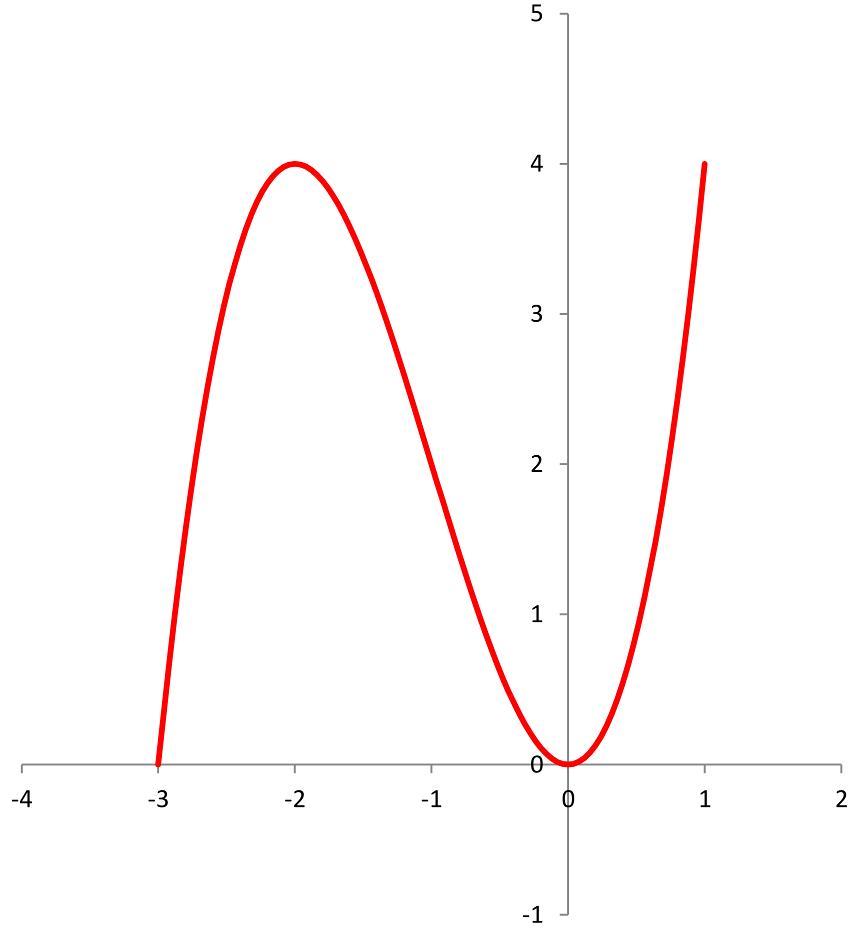
1. $x = 0$

   Local minimum

2. $x = -2$

   Local maximum

No absolute extrema



The First Derivative Test also applies for finding relative extrema at points where f is not differentiable (as long as f is continuous at the point).

Example 1b

$$f(x) = x^{2/3}(x-5)$$

$$f'(x) = \frac{5}{3}x^{2/3} - \frac{10}{3}x^{-1/3}$$

Critical values at

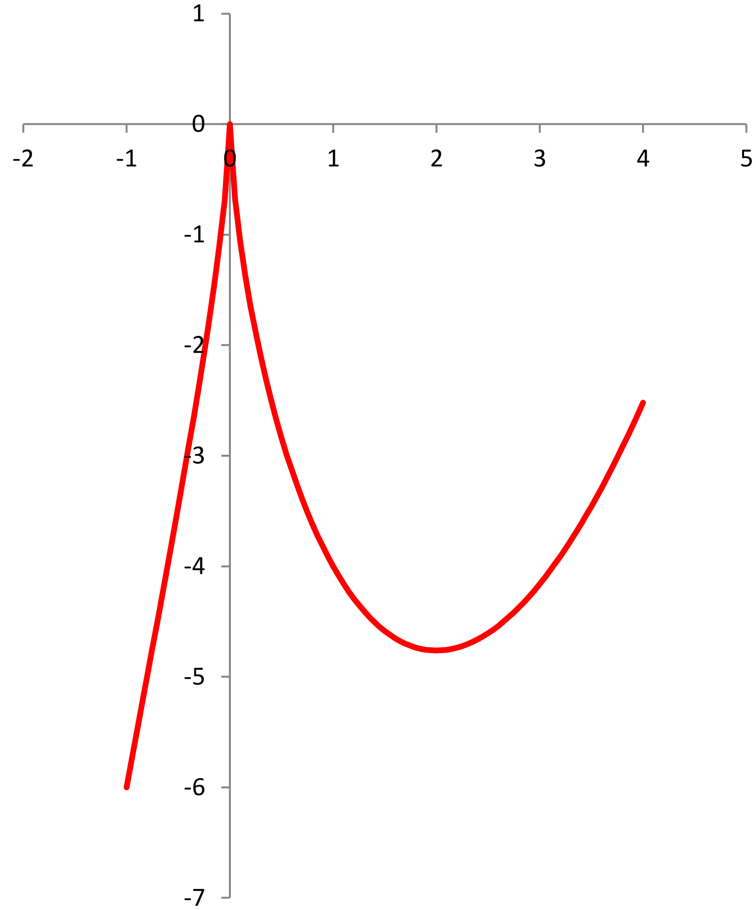
1. $x = 0$

   Local maximum

2. $x = 2$

   Local minimum

No absolute extrema



Exercise 1

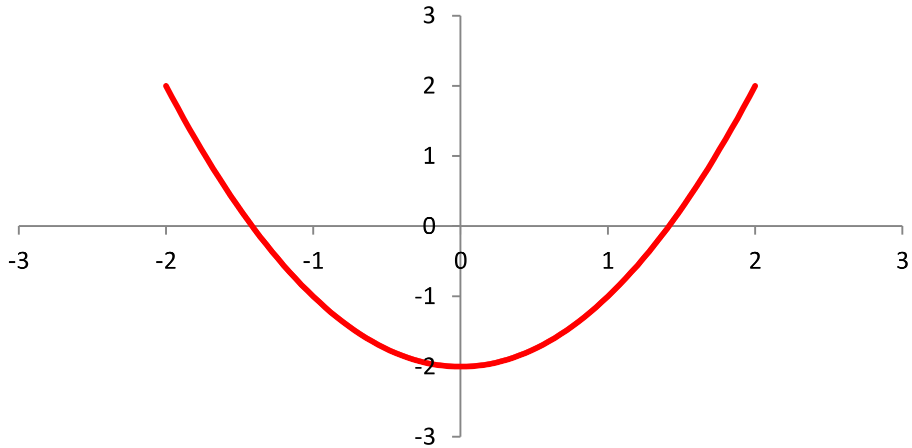
Use the First Derivative Test to find the local extreme values. Identify any absolute extrema.

1. $f(x) = x^3 - 6x^2 + 15$

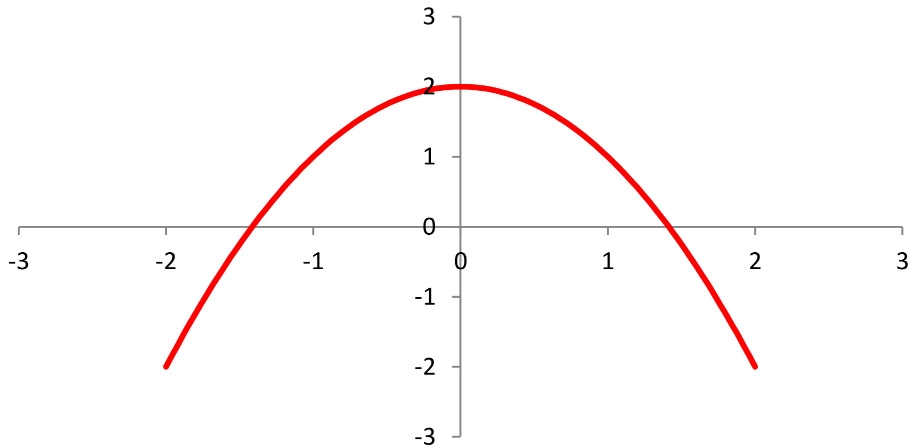
2. $x^{2/3} - 4$

Concavity

Concave up

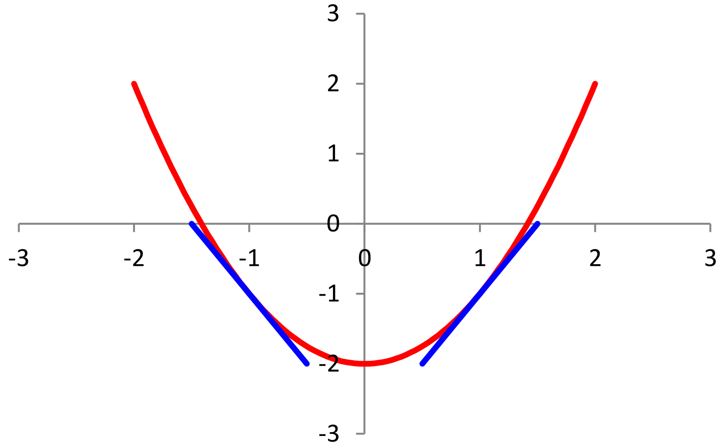


Concave down

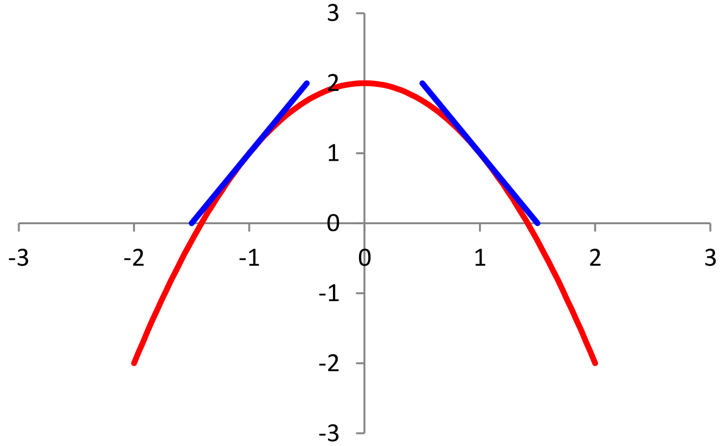


Concavity of a Function

- f is concave up on an interval if f' is increasing on that interval.



- f is concave down on an interval if f' is decreasing on that interval.



Concavity Test

- Just as $f' > 0$ on intervals where f is increasing, $f'' > 0$ on intervals where f' is increasing.
- Likewise, $f'' < 0$ on intervals where f' is decreasing.
- If $f'' > 0$ on an interval, then f is concave up on that interval.
- If $f'' < 0$ on an interval, then f is concave down on that interval.

Example 2

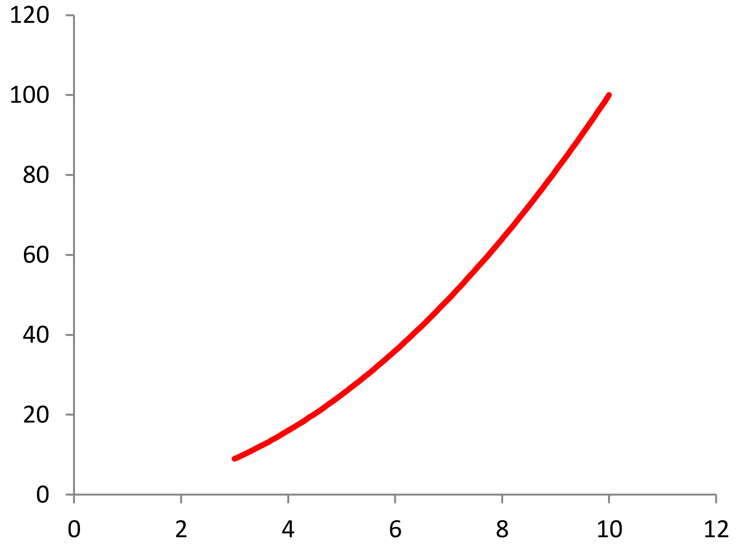
Use the Concavity Test to determine the concavity of the given function on the given interval.

a. $f(x) = x^2$

$(3, 10)$

$f''(x) = 2$

Concave up

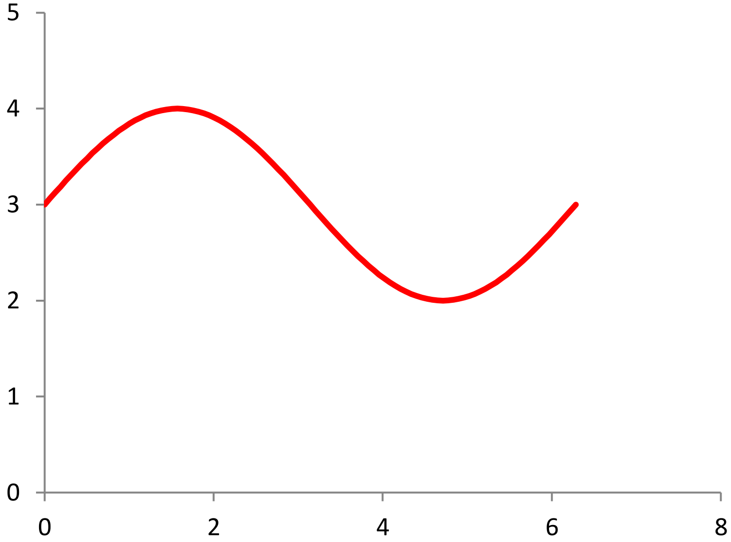


Example 2

Use the Concavity Test to determine the concavity of the given function on the given interval.

b. $f(x) = 3 + \sin x$

$(0, 2\pi)$



$f''(x) = -\sin x$

Concave down $(0, \pi)$

Concave up $(\pi, 2\pi)$

Exercise 2

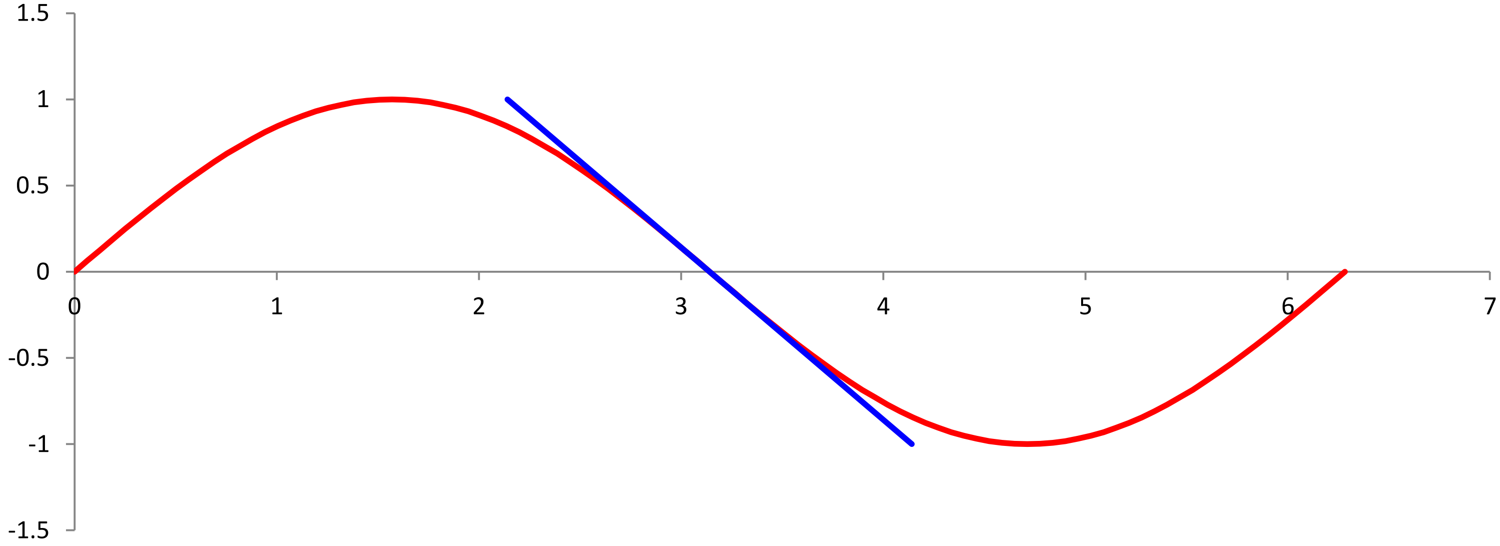
Use the Concavity Test to determine the intervals on which the graph of $y = -x^4 + 4x^3 - 4x + 1$ is

a. concave up

b. concave down

Inflection Points

A point where the graph of a function has a tangent line and where the concavity changes is a point of inflection.

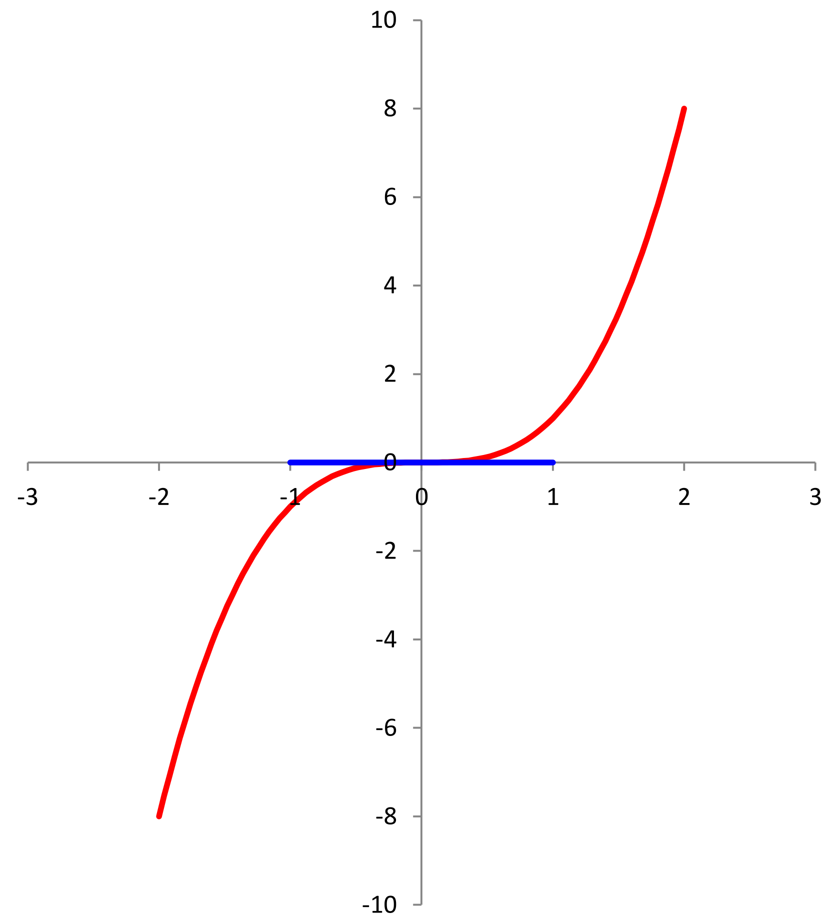


Find the Inflection Point(s) of a Function

1. Find the critical points $x = c$ by finding the x-values where $f''(x) = 0$ or $f''(x)$ is undefined.
2. If $f''(x)$ changes from positive to negative, or from negative to positive at $x = c$ and f is continuous at $x = c$, then $\left(c, f(c)\right)$ is an inflection point of f.

Example 3

- $f(x) = x^3$ on $[-2, 2]$
- $f'(x) = 3x^2 \Longrightarrow$ $f''(x) = 6x$
- $6x = 0 \Longrightarrow x = 0$ is a critical number.
- $(0, 0)$ is an inflection point of f.

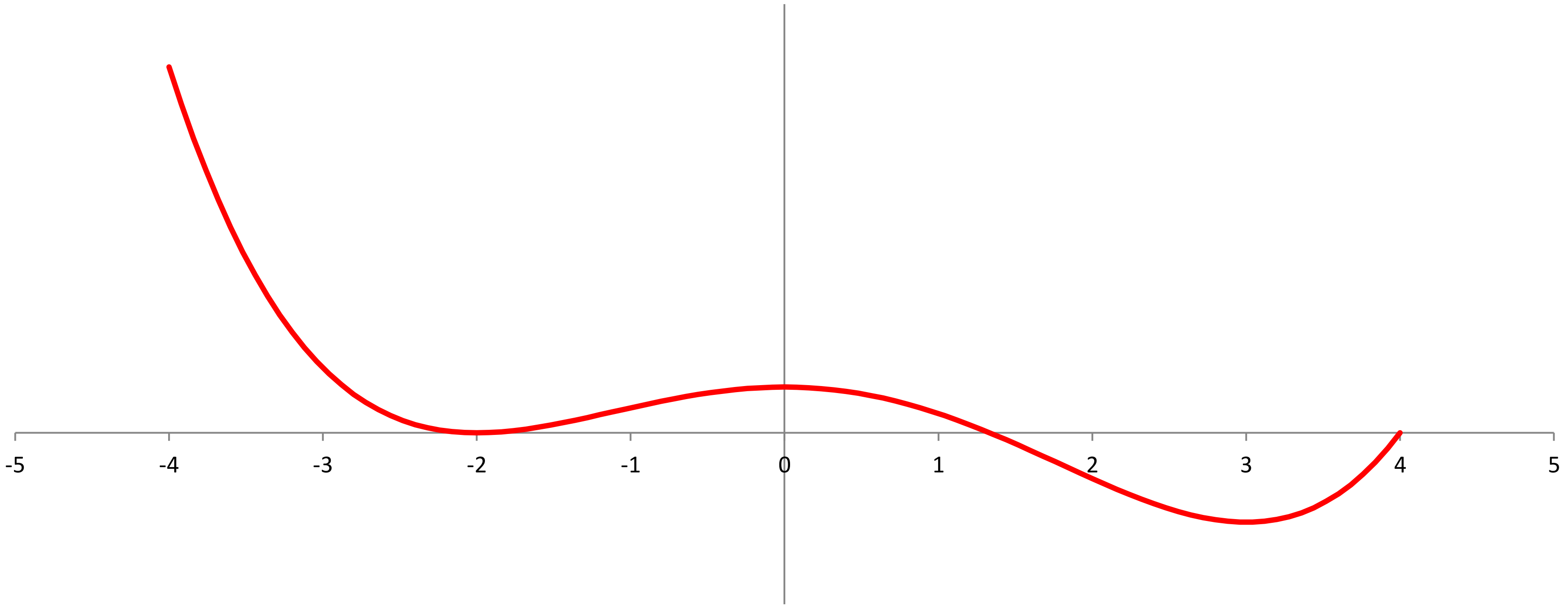


Exercise 3

Find all the points of inflection on the graph of $y = e^{-x^2}$.

Example 4

The graph of the *derivative* of a function f is shown below.



a. On what interval is f increasing?

   Since $f' > 0$ on the intervals [–4, –2) and $(-2, 4/3)$, f is increasing on $[-4, 4/3]$ with a horizontal tangent at $x = -2$ (a “shelf point”).

b. On what intervals is f concave up?

   f is concave up where f' is increasing: (–2, 0) and (3, 4).

c. At which *x*-coordinates does f have local extrema?

Since f' changes from positive to negative at $x = 1$, f must have a local maximum.

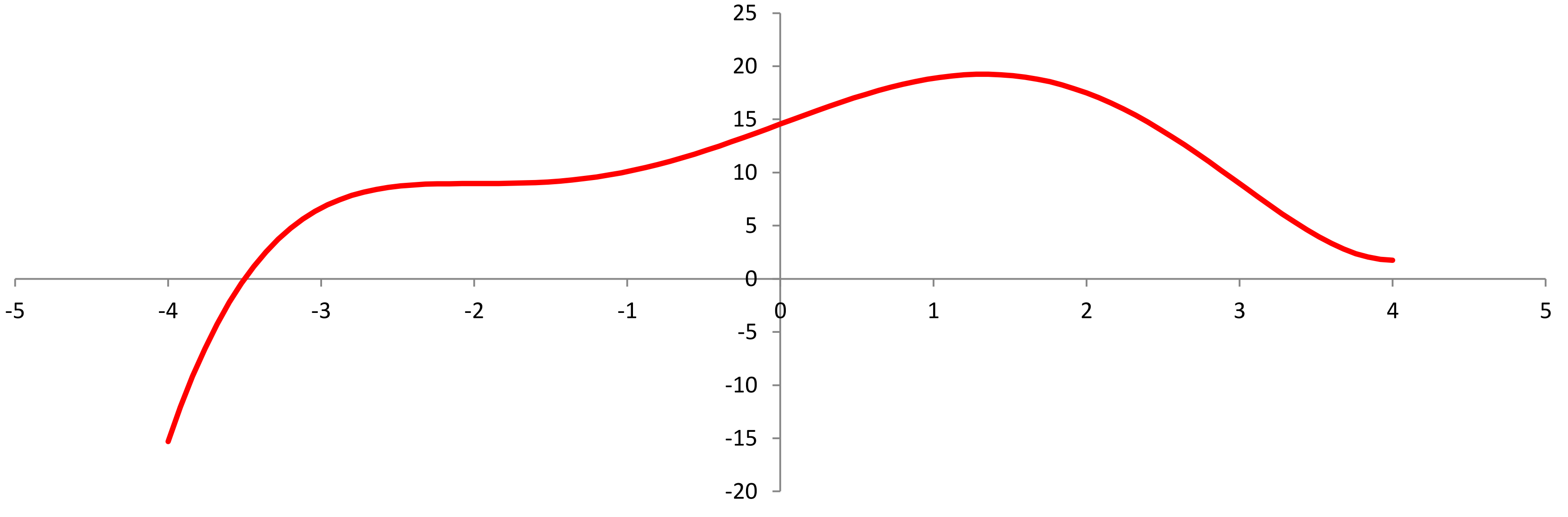
Since f does not change sign at $x = -2$, there is no local extremum there.

Since the function increases from the left end point and decreases from the right end point, there are local minima at the end points.

d. What are the *x*-coordinates of all the inflection points of the graph of f?

The inflection points of f are the same as the turning points of f': -2, 0, and 3.

e. Sketch a possible graph of f.



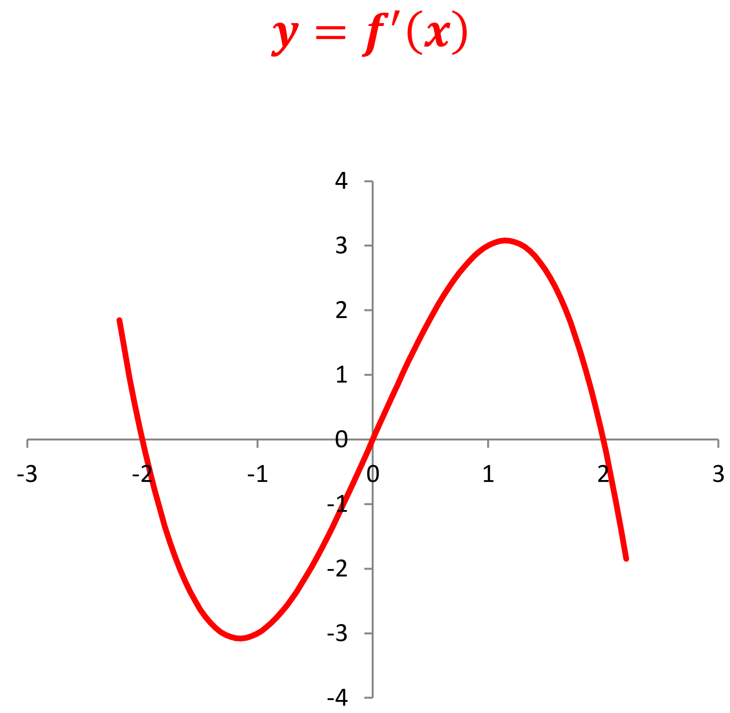
Exercise 4

Use the graph of f' to estimate,

a. The intervals on which f is increasing.

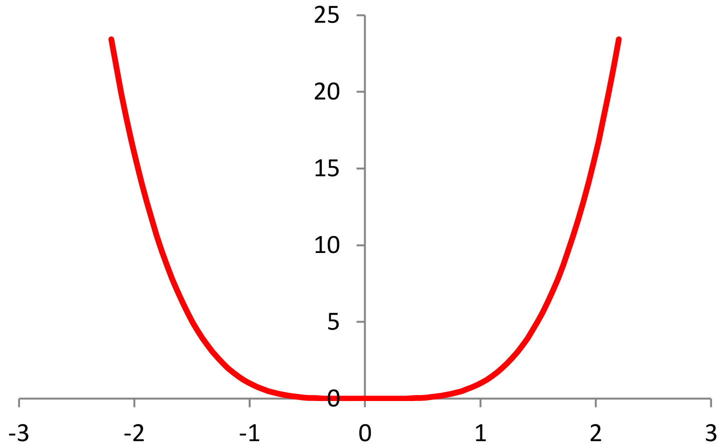
b. The intervals on which f is decreasing.

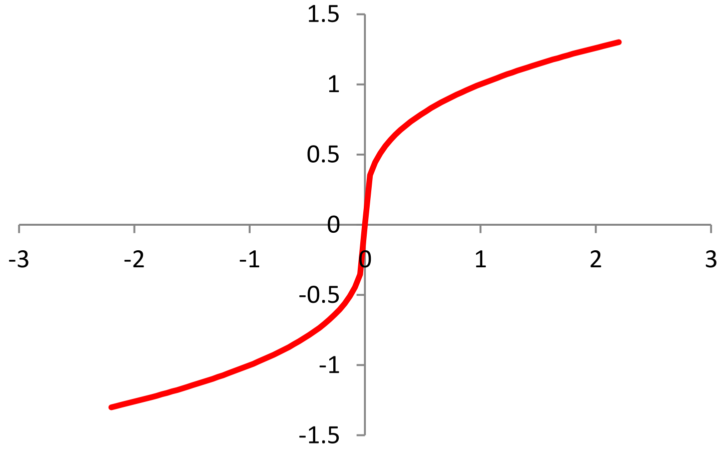
c. The x-coordinates of all local extreme values.



Caution

- The second derivative can be 0 at a noninflection point.
  $$f(x) = x^4$$
- The second derivative need not be 0 at an inflection point.
  $$f(x) = \sqrt[3]{x}$$





Example 5

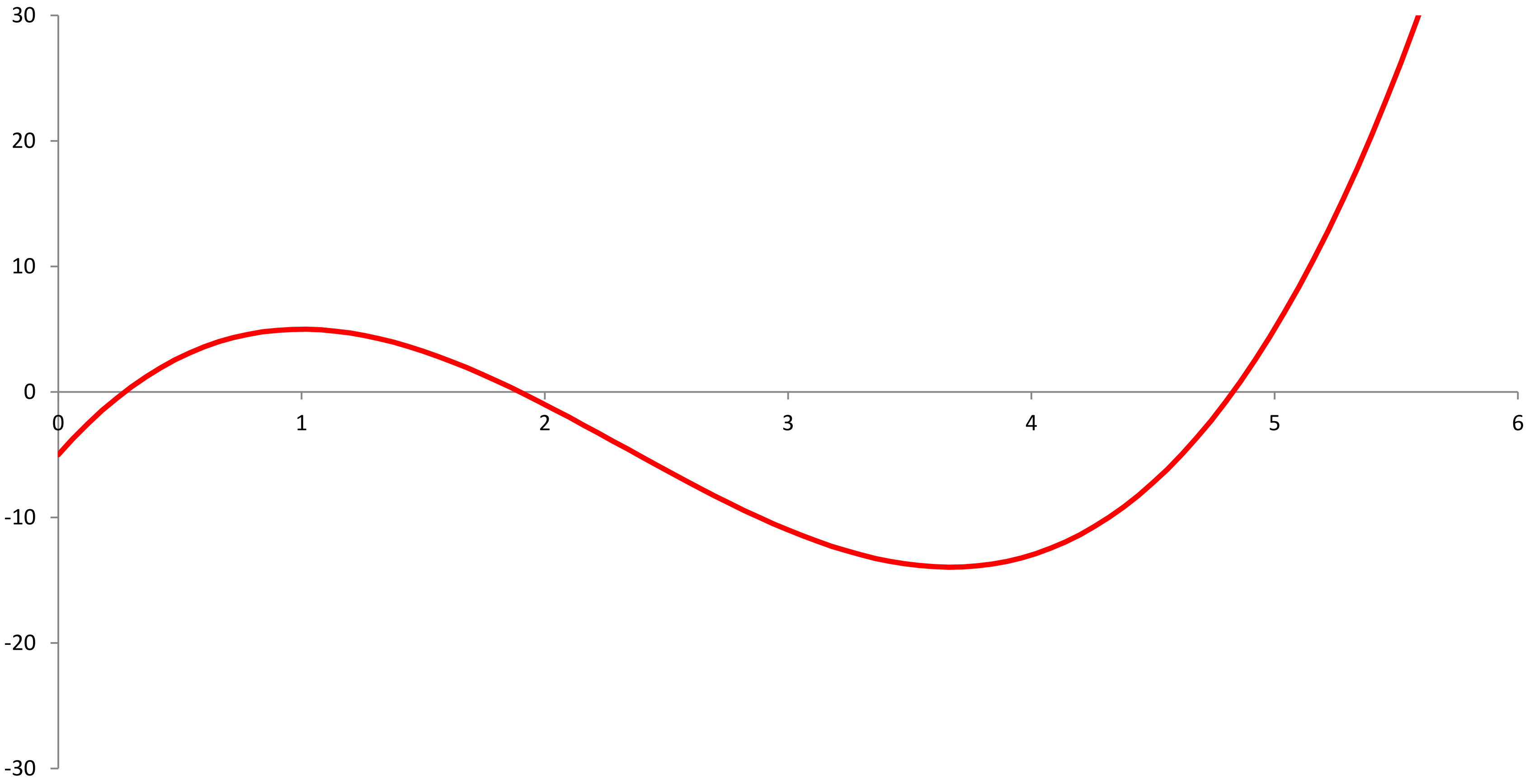
A particle is moving along the *x*-axis with position function

$$x(t) = 2t^3 - 14t^2 + 22t - 5, t \geq 0$$

When $x(t)$ is increasing, the particle is moving to the right.

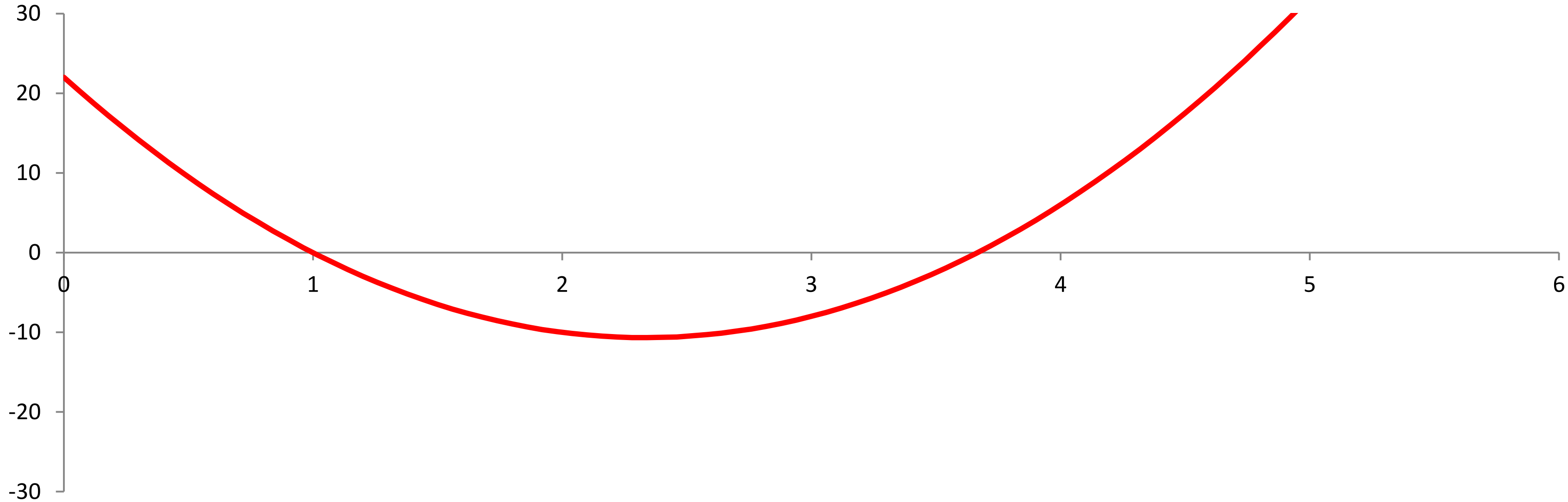
When $x(t)$ is decreasing, the particle is moving to the left.

Position Curve



1. Find the velocity

$$v(t) = x'(t) = 6t^2 - 28t + 22$$

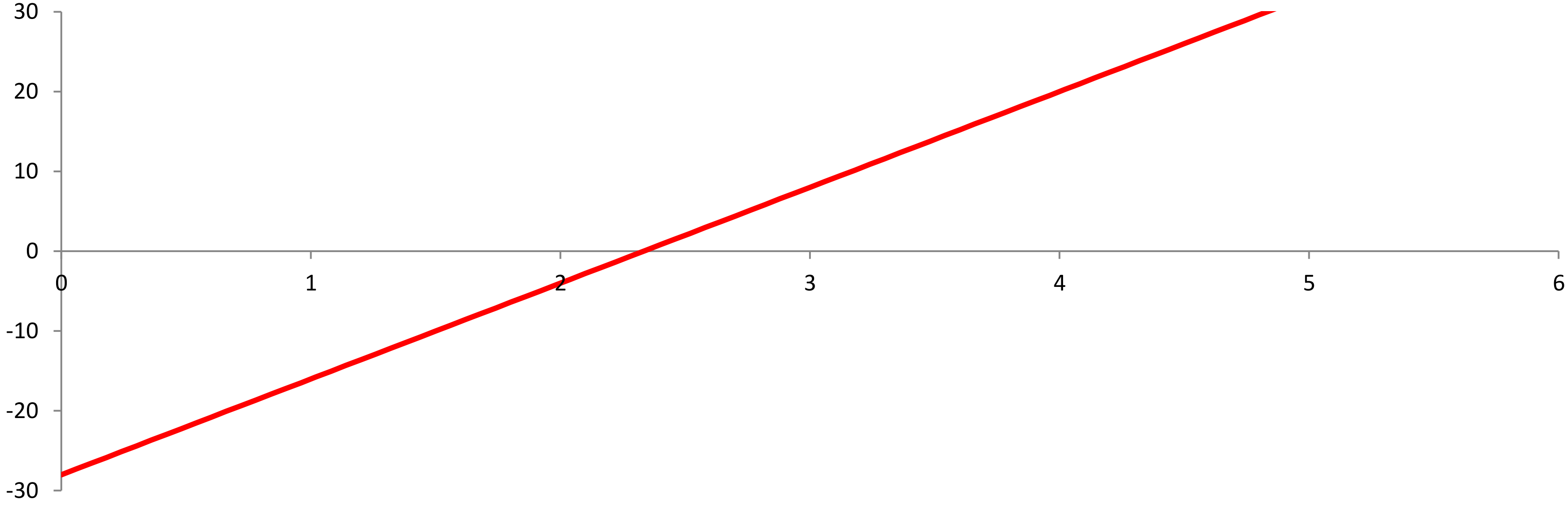


$$x'(t) = 0 \text{ when } t = 1 \text{ and } t = 11/3$$

Interval	Sign of x'	Behavior of x	Particle motion
$(0, 1)$	$+$	increasing	right
$(1, 11/3)$	$-$	decreasing	left
$(11/3, \infty)$	$+$	increasing	right

2. Find the acceleration

$$a(t) = v'(t) = x''(t) = 12t - 28$$



Interval	Sign of x''	Graph of x	Particle motion
$[0, 7/3]$	$-$	concave down	decelerating
$[7/3, \infty)$	$+$	concave up	accelerating

Exercise 5

A particle is moving along the *x*-axis with position function

$$x(t) = t^2 - 4t + 3, \qquad t \geq 0$$

a. Find the velocity.

b. Find the acceleration.

c. Describe the motion of the particle for $t \geq 0$.

Second Derivative Test for Relative Extrema

- If $f'(c) = 0$ and $f''(x)$ exists on an open interval containing c,

1. If $f''(c) > 0$, then $f(c)$ is a relative minimum.
2. If $f''(c) < 0$, then $f(c)$ is a relative maximum.
3. If $f''(c) = 0$, then there is no conclusion.

Example 6

- $f(x) = 5 + 3x^2 - x^3$
- $f'(x) = 6x - 3x^2$
  - Critical values
    - $x = 0, 2$
    - $x = 0, 2$

Example 6

$f''(x) = 6 - 3x$

1. verify relative minimum at $x = 0$

   $f''(0) = 6 - 3 \cdot 0 > 0$

2. verify relative maximum at $x = 2$.

   $f''(0) = 6 - 3 \cdot 0 > 0$

Exercise 6

Use the Second Derivative Test to fine the local extrema for,

$$y = x^3 + 3x^2 - 2$$

Homework

P 215: 1-11 odd, 15-30 multiples of 3, 44, 48