Chapter 4.2

Mean Value Theorem

Objective

Apply the Mean Value Theorem and find the intervals on which a function is increasing or decreasing.

Learning Target

80% of the students will be able to find the antiderivative of the following function:

$$f(t) = 9.8t$$

Standard

F-IF.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

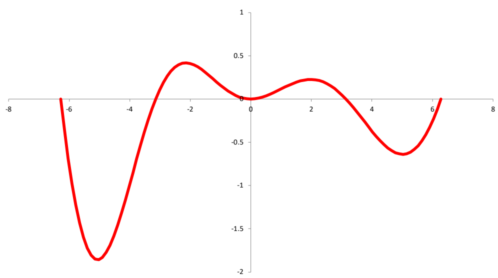
Overview

- Mean Value Theorem.
- Physical Value Interpretation.
- Increasing and Decreasing Functions.
- Other Consequences.

Rolle's Theorem

- Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). If $f(a) = f(b)$, then there is at least one number c in (a, b) such that $f'(c) = 0$.
- This means that if the graph starts and ends at the same y-value, then there must be at least one relative maximum or relative minimum in (a, b), or the function is constant.

Illustrating Rolle's Theorem



Mean Value Theorem

Rolle's Theorem is a special case of the Mean Value Theorem.

Mean Value Theorem for Derivatives

If $y = f(x)$ is continuous on every point of the closed interval $[a, b]$ and differentiable at every point of its interior (a, b), then there is at least one point in (a, b) at which

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

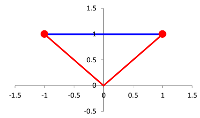
Mean Value Theorem

- $f'(c)$ is the instantaneous rate of change of f at $x = c$.
- $\frac{f(b)-f(a)}{b-a}$ is the average rate of change of f over the interval $[a, b]$.

Mean Value Theorem

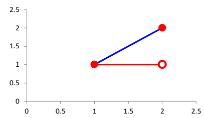
$f(x) = |x| \quad [-1, 1]$

Not differentiable at $x = 0$.



$f(x) = \begin{cases} 1, 1 \le x < 2 \\ 2, \quad\quad x = 2 \end{cases}$

Not continuous at $x = 2$.



Example 1

Let $f(x) = x^2$ on $[0, 2]$.

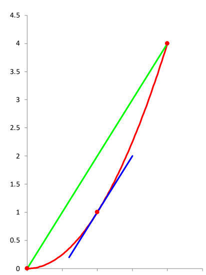
$f(x)$ is continuous on $[0, 2]$.

$f(x)$ is differentiable on $(0, 2)$.

$$f'(c) = \frac{f(2) - f(0)}{2 - 0} = 2$$

$f'(x) = 2x$

$c = 2$



Exercise 1

Let $f(x) = x^2 + 2x - 1$ on $[0, 1]$.

a. Explain why the function satisfies the hypotheses of the Mean Value Theorem on $[0, 1]$.

b. Find each value of c in $(0, 1)$ that satisfies the equation

$$f'(c) = \frac{f(2) - f(1)}{2 - 1}$$

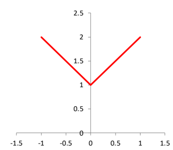
Example 2a

Let $f(x) = \sqrt{x^2} - 1$ on $[-1, 1]$.

$f(x)$ is continuous on $[-1, 1]$.

$f'(0)$ is undefined.

Therefore, the Mean Value Theorem does not apply.



Example 2b

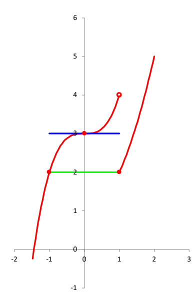
Let $f(x) = \begin{cases} x^3 + 3 \Leftarrow x < 1 \\ x^2 + 1 \Leftarrow x \geq 1 \end{cases}$ on $[-1, 1]$.

$f(x)$ is discontinuous at $x = 1$.

Therefore, the Mean Value Theorem does not apply.

Nevertheless if $x = 0$,

$$\frac{d}{dx}(x^3 + 3) = \frac{f(1) - f(-1)}{1 - (-1)}$$



Converse of the Mean Value Theorem

- The converse of the Mean Value Theorem is not necessarily true.

Exercise 2

Let $f(x) = \sin^{-1} x$ on $[-1, 1]$.

a. Does f satisfy the hypotheses of the Mean Value Theorem on $[-1, 1]$?

b. If it does, find each value of c in $(-1, 1)$ that satisfies the equation

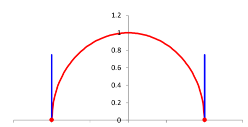
$$f'(c) = \frac{f(1) - f(-1)}{1 - (-1)}.$$

Example 3

Let $f(x) = \sqrt{1 - x^2}$ on $[-1, 1]$.

Find c:

$$f'(c) = \frac{f(1) - f(-1)}{1 - (-1)}.$$



f is continuous on $[-1, 1]$.

f' is undefined at $x = \pm 1$, but

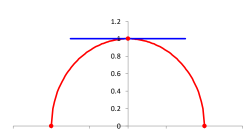
$$f'(x) = \frac{-x}{\sqrt{1 - x^2}} \text{ on } (-1, 1).$$

Example 3

$$f'(c) = \frac{-c}{\sqrt{1 - c^2}} = \frac{f(-1) - f(1)}{1 - (-1)} = 0$$

$$c = 0, \qquad f(0) = 1$$

$$y - 1 = 0(x - 0) \qquad y = 1$$



Exercise 4

Let $f(x) = \sqrt{x-1}$ on [1, 3].
$A = (1, f(1)), \quad B = (3, f(3))$.
Write an equation for

a. the secant line AB.
b. a tangent line to f in the interval $(1, 3)$ that is parallel to AB.

Physical Interpretation

The average change of in f over some interval $[a, b]$ is the difference quotient

$$\frac{f(b) - f(a)}{b - a}.$$

$f'(c)$ is the instantaneous change of f at some interior point c in the interval (a, b).

The Mean Value Theorem tells us that the instantaneous change at some interior point must equal the average change over the entire interval.

Example 5

- The Mean Value Theorem can be used to prove that someone is speeding between two checkpoints. Suppose that the speed limit is 60 mph. A car enters the Mass Pike and takes a ticket. The car exits the highway 20 miles later and the official punches the ticket. If the time punched is 18 minutes later than the entrance time, the driver must have exceeded the speed limit at some point.

Example 5

- Let $s(t)$ represent the position of the car.
- Average velocity $= \frac{s(b)-s(a)}{b-a} = \frac{s\left(\frac{18}{60}\right)-s(0)}{\frac{18}{60}-0}$

$= \frac{20}{\frac{18}{60}} = \frac{200}{3} \approx 66.7$ mph.

- By the Mean Value Theorem, there exists a time t such that $f'(t) = 66.7$, where $f'(t)$ is the instantaneous velocity at t.

Exercise 5

A trucker handed in a ticket at a toll booth showing that in 2 hours she had covered 159 miles on a toll road with a speed limit of 65 mph. The trucker was cited for speeding. Why?

Increasing and Decreasing Functions

- A function f is <u>increasing</u> on an interval if for any two numbers x_1 and x_2 in the interval, if $x_1 < x_2$ *then* $f(x_1) < f(x_2)$.
- A function f is <u>decreasing</u> on an interval if for any two numbers x_1 and x_2 in the interval, if $x_1 < x_2$ *then* $f(x_1) > f(x_2)$.

Increasing and Decreasing Functions

- Let f be a function that is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b).
- If $f'(x) > 0$ for all x in (a, b), then f is increasing on $[a, b]$.
- If $f'(x) < 0$ for all x in (a, b), then f is decreasing on $[a, b]$.
- If $f'(x) = 0$ for all x in (a, b), then f is constant on $[a, b]$.

Determining Where Graphs Rise or Fall

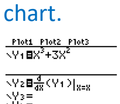
- To find the intervals where a function f is increasing or decreasing,

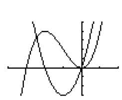
1. Find the critical numbers of f.
2. Draw a labeled sign chart for f'.

Example 6

- Let $f(x) = x^3 + 3x^2$
- $f'(x) = 3x^2 + 6x = 0 \Longrightarrow x = 0, x = -2.$
- Draw a labeled sign chart.

Interval	Sign of f'
$(-\infty, -2)$	$+$
$(-2, 0)$	$-$
$(0, \infty)$	$+$





- f is increasing on $(-\infty, -2) \cup (0, \infty)$
- f is decreasing on $(-2, 0)$

Exercise 6

Let $f(x) = \frac{-x}{x^2+4}$

a. Find the local extrema.
b. Find the intervals on which the function is increasing.
c. Find the intervals on which the function is decreasing.

Antiderivative

- $F(x)$ is an antiderivative of $f(x)$ if $F'(x) = f(x)$
- $F(x) = x^2$ is an antiderivative of $f(x) = 2x$, since $F'(x) = 2x$
- $F(x) = x^2 + 5$ is also an antiderivative of $f(x) = 2x$, since $F'(x) = 2x$
- We have a family of antiderivatives for the function $f(x) = 2x$ denoted by $F(x) = x^2 + C$ where C is any real number.

Constants in Functions

- If $f'(x) = 0$ on some interval I, then $f(c) = C$ on I, where C is constant.
- If $f'(x) = g'(x)$ then $f(x) = g(x) + C$, where C is constant.

Example 7

Find the function $f(x)$, when $f'(x) = \sin x$, and $f(0) = 2$.

$f(x) = -\cos x + C$

The graph of f passes through the point (0, 2).

$2 = -\cos 0 + C$

$C = 2 + \cos 0 = 2 + 1 = 3$

$f(x) = 3 - \cos x$

Exercise 7

Let $f'(x) = -\frac{1}{x^2}$, $x > 0$, and f passes through the point (2, 1). Find $f(x)$.

Example 8

Find the velocity and position functions for a body falling freely from a height of 0 meters with an acceleration of 9.8 m/s^2:

a. The body falls from rest.

The acceleration is the constant function $a = 9.8$.

The velocity is the antiderivative of the acceleration $v(t) = 9.8t + C$.

Since the body falls from rest, $v(0) = 0$.

$\therefore\ C = 0$.

The body's velocity function is, $v(t) = 9.8t$.

The body's position function is the antiderivative of the velocity function.
$s(t) = 4.9t^2 + C$.

Since $s(0) = 0$, $C = 0$.

The body's position function is, $s(t) = 4.9t^2$.

b. The body is propelled with an initial downward velocity of 1 m/s.

The acceleration is the constant function $a = 9.8$.

The velocity is the antiderivative of the acceleration $v(t) = 9.8t + C$.

Since the body has an initial velocity of 1 m/s, $v(0) = 1$.

$\therefore\ C = 1$.

The body's velocity function is, $v(t) = 9.8t + 1$.

The body's position function is the antiderivative of the velocity function.
$s(t) = 4.9t^2 + t + C$.

Since $s(0) = 0$, $C = 0$.

The body's position function is, $s(t) = 4.9t^2 + t$.

Exercise 8

On the moon, the acceleration due to gravity is $1.6\,m/s^2$.

a. If a rock is dropped into a crevasse, how fast will it be going just before it hits bottom 30 s later?
b. How far below the point of release is the bottom of the crevasse?
c. If the rock is thrown into the crevasse from the same point, with a downward velocity of 4 m/s, when will it hit the bottom, and how fast will it be going?

Homework

P 202: 3-33 multiples of 3, 37, 43, 45, 48, 58