Chapter 4.1

Extreme Values of Functions

Objective

Determine the local and extreme values of a function.

Learning Target

80% of the students will be able to find the extreme values of the following function both graphically and analytically:

$$f(x) = \frac{1}{|x^2+1|}, \qquad -2 \leq x \leq 2.$$

Standard

F-IF.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

Overview

- Absolute (Global) Extreme Values.
- Local (Relative) Extreme Values.
- Finding Extreme Values.

Absolute (Global) Extreme Values

Knowing the extreme values of a function can provide critical information.

- What is the most efficient way to manufacture a product (minimize cost)?
- How to achieve the maximum production.

Absolute Extrema

- The absolute minimum and absolute maximum values of a function over an interval are also called the absolute extrema of the function over the interval.

Absolute Extrema

- $f(c)$ is the absolute maximum value of $f(x)$ in an interval if $f(c) \geq f(x)$ for all x in the interval (there could be more than one x-value where this occurs).
- $f(c)$ is the absolute minimum value of $f(x)$ in an interval if $f(c) \leq f(x)$ for all x in the interval (there could be more than one x-value where this occurs).

Absolute Extreme Values

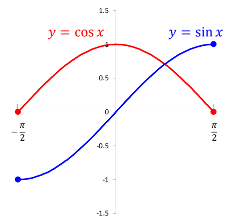
Let f be a function with domain D. Then $f(c)$ is the

a. absolute maximum value on D if and only if $f(x) \leq f(c)$ for all x in D.

b. absolute minimum value on D if and only if $f(x) \geq f(c)$ for all x in D.

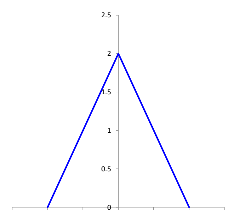
Example 1
Absolute Extrema

- $y = \cos x$
  - Maximum: $x = 0$
  - Minimum: $x = \pm\frac{\pi}{2}$
- $y = \sin x$
  - Maximum: $x = \frac{\pi}{2}$
  - Minimum: $x = -\frac{\pi}{2}$



Exercise 1

Find the extreme values and where they occur.



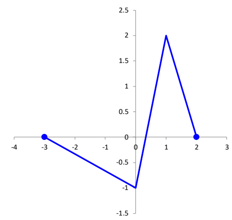
Example 2
Exploring Absolute extrema

Find the absolute extrema for the function $y = x^2$ given the following domains:

$(-\infty, \infty)$	No absolute maximum. Absolute minimum of 0 at $x = 0$.
$[0,2]$	Absolute maximum of 4 at $x = 2$. Absolute minimum of 0 at $x = 0$.
$(0, 2]$	Absolute maximum of 4 at $x = 2$. No absolute minimum.
$(0,2)$	No absolute extrema.

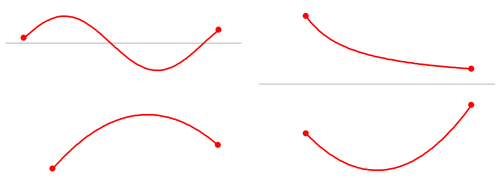
Exercise 2

Find the absolute extreme values and where they occur.

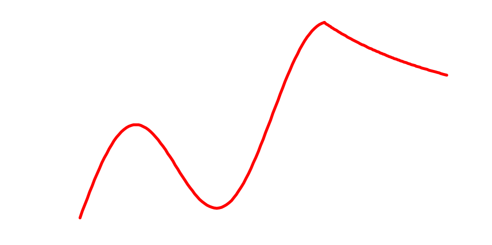


Extreme Value Theorem

If f is continuous on the closed interval $[a, b]$ then f has both a maximum value and a minimum value on the interval.



Local Extreme Values



Local Extreme Values

- If there is an open interval (no matter how small) containing $x = c$ on which $f(c)$ is a maximum, then $f(c)$ is called a local maximum of f.
- If there is an open interval (no matter how small) containing $x = c$ on which $f(c)$ is a minimum, then $f(c)$ is called a local minimum of f.

Local Extreme Values

Let c be an interior point of the domain of the function f. Then $f(c)$ is a

a. **local maximum value** at c if and only if $f(x) \leq f(c)$ for all x in some open interval containing c.

b. **local minimum value** at c if and only if $f(x) \geq f(c)$ for all x in some open interval containing c.

Local Extreme Values

Notice that a function f has a local maximum or a local minimum at *an endpoint* c if the appropriate inequality holds for all x in some half-open domain interval containing c.

Absolute Extreme Values

Local extrema are also called **relative extrema**.

An absolute extremum is also a local extremum, because being an extreme value overall makes it an extreme value in the immediate neighborhood.

A list of local extrema will include absolute extrema, if there are any.

Local Extreme Values

- Local extrema occur at cusps (where $f'(c)$ is undefined) or where there is a horizontal tangent line (where $f'(c) = 0$). f must be continuous at $x = c$.

Local Extreme Values

If a function f has a local maximum value or a local minimum value at an interior point c of its domain, and if f' exists at c, then

$$f'(c) = 0.$$

Finding Local Extrema Analytically

- The possible x-values where local extrema could occur are called critical numbers.
- The point (x, y) is called a critical point.
- To find the critical numbers of $f(x)$
  1. Find $f'(x)$.
  2. Find the x-values where $f'(x) = 0$ or where $f'(x)$ is undefined (need to verify continuity).

Finding Absolute Extrema Analytically

To find the absolute maximum and/or absolute minimum value of f on a closed interval $[a, b]$, **make a chart** listing the critical numbers of f and the endpoints of the interval.

Finding Absolute Extrema Analytically

- The absolute minimum is the smallest $f(x)$ value (could be a tie).
- The absolute maximum is the largest $f(x)$ value (could be a tie).
- The absolute extrema must occur at an endpoint or at a critical number.

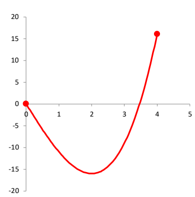
Example 3

- Find the absolute extrema of $f(x) = x^3 - 12x$ on $[0, 4]$.
- Critical numbers:
  - $f'(x) = 3x^2 - 12$
  - $3x^2 - 12 = 0 \Longrightarrow 3x^2 = 12 \Longrightarrow x^2 = 4$
  - $x = \pm 2$ (only $x = 2$ is in the interval $[0, 4]$.)

Example 3

x	$f(x)$
2	-16
0	0
4	16

- -16 is the absolute minimum.
- 16 is the absolute maximum
- Check graphically.



Exercise 3

Find the absolute extrema of of the following function on the closed interval [–1, 2]:
$f(x) = x^3 - \frac{3}{2}x^2$

Make a table. Then confirm your answer using a graphing calculator.

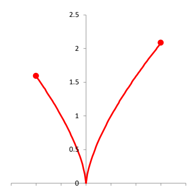
Example 4

- Find the absolute extrema of $f(x) = x^{2/3}$ on $[-2, 3]$.
- Critical numbers:
  - $f'(x) = \frac{2}{3}x^{-1/3}$
  - No zeros, but $f'(0)$ is undefined.

Example 4

x	$f(x)$
-2	$\sqrt[3]{4}$
0	0
3	$\sqrt[3]{9}$

- 0 is the absolute minimum.
- $\sqrt[3]{9}$ is the absolute maximum
- Check graphically.



Exercise 4

Find the absolute extrema of of the following function on the closed interval $[-1, 2]$:
$f(x) = 3x^{2/3} - 2x$

Make a table. Then confirm your answer using a graphing calculator.

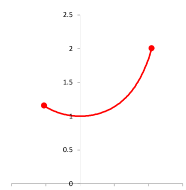
Example 5

- Find the absolute extrema on $\left[-\frac{\pi}{6}, \frac{\pi}{3}\right]$ of $g(x) = \sec x$.
- Critical numbers:
  - $g'(x) = \sec x \tan x = \frac{\sin x}{\cos^2 x}$
  - $\frac{\sin x}{\cos^2 x} = 0$ when $\sin x = 0 \Longrightarrow x = 0$ in $\left[-\frac{\pi}{6}, \frac{\pi}{3}\right]$;
  - $\frac{\sin x}{\cos^2 x}$ is undefined when $\cos x = 0$, but $x = \pm\frac{\pi}{2}$ not in $\left[-\frac{\pi}{6}, \frac{\pi}{3}\right]$;

Example 5

x	$f(x)$
$-\frac{\pi}{6}$	$\frac{2}{\sqrt{3}} \approx 1.155$
0	1
$\frac{\pi}{3}$	2

- 1 is the absolute minimum.
- 2 is the absolute maximum
- Check graphically.



Exercise 5

Find the absolute extrema of of the following function on the closed interval [−1, 6]:
$h(t) = \frac{t}{t+3}$

Make a table. Then confirm your answer using a graphing calculator.

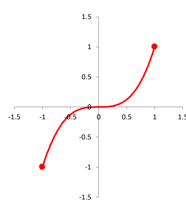
Example 6

- Not every critical point indicates an extremum.
- Find the absolute extrema of $g(x) = x^3$ on $[-1, 1]$.
- Critical numbers:
  - $g'(x) = 3x^2$
  - $3x^2 = 0$ when $x = 0$.

Example 6

x	$f(x)$
-1	-1
0	0
1	1

- -1 is the absolute minimum.
- 1 is the absolute maximum
- Check graphically.



Exercise 6

Find the absolute extrema of of the following function on the closed interval [–1, 1]:
$f(x) = \sqrt[3]{x}$

Make a table. Then confirm your answer using a graphing calculator.

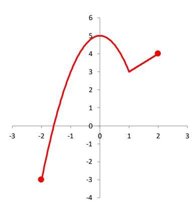
Example 7

- Find the absolute extrema on $[-2, 2]$ of $g(x) = \begin{cases} 5 - 2x^2, x \leq 1 \\ x + 2, \quad x > 1 \end{cases}$.
- Critical numbers:
  - $x \leq 1 \Rightarrow g'(x) = -4x$
    - $g'(0) = 0$
    - $\lim_{x\to 1^-} g'(x) \neq \lim_{x\to 1^+} g'(x)$
      - $g'(1)$ is undefined.
  - $x > 1 \Rightarrow g'(x) = 1 \neq 0$.

Example 7

x	$f(x)$
-2	-3
0	5
1	3
2	4

- -3 is the absolute minimum.
- 5 is the absolute maximum
- Check graphically.



Exercise 7

Find the absolute extrema of of the following function on the closed interval [−2, 2]:

$$g(x) = \begin{cases} -x^2 - 2x + 4, & x \leq 1 \\ -x^2 + 6x - 4, & x > 1 \end{cases}$$

Make a table. Then confirm your answer using a graphing calculator.

Example 8
Using Graphical Methods

Find the extreme values of

$$f(x) = \ln\left|\frac{x}{1+x^2}\right|$$

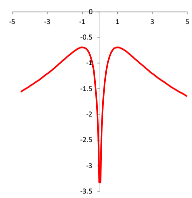
Domain: $x \neq 0$

Graph the function.

Appears to have absolute maxima of –0.7 at ±1

No absolute minima

Use your graphing calculator.



Analytical Methods

$$f'(x) = \frac{1-x^2}{x(1+x^2)}$$

$f'(x) = 0 \Leftarrow x = \pm 1$

$f'(x)$ defined for all points on domain.

$$f(\pm 1) = \ln\frac{1}{2} = -\ln 2 \approx -0.693$$

$f(0)$ is undefined.

Exercise 8

Identify all critical points and determine the local extreme values for the function,

$$y = x^2\sqrt{3-x}$$

Homework

P 193: 3-30 multiples of 3, 35, 37, 40, 43, 54