Chapter 3.9

Derivatives of Exponential and Logarithmic Functions

Objectives

- Calculate derivatives of exponential and logarithmic functions.

Learning Target

- 80% of the students will be able to calculate the derivative of x^{-6x}.

Standard

G-C.4 Construct a tangent line from a point outside a given circle to the circle.

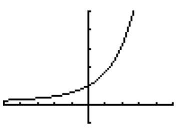
Overview

- Derivatives of e^x
- Derivative of a^x
- Derivative of $\ln x$
- Derivative of $\log_a x$
- Power rule for arbitrary real power

Essential Limit

Find

$$\lim_{h \to 0} \frac{e^h - 1}{h}$$



X	Y_1
-.03	.98515
-.02	.99007
-.01	.99502
0	ERROR
.01	1.005
.02	1.0101
.03	1.0152

Press + for ΔTbl

Derivative of e^x

$$\frac{d}{dx}(e^x) = \lim_{h\to 0}\frac{e^{x+h}-e^x}{h}$$
$$= \lim_{h\to 0}\frac{e^x\cdot e^h - e^x}{h}$$
$$= \lim_{h\to 0}\left(e^x\cdot\frac{e^h-1}{h}\right)$$
$$= e^x\cdot\lim_{h\to 0}\frac{e^h-1}{h}$$

Derivative of e^x

$$\frac{d}{dx}(e^x) = e^x\cdot\lim_{h\to 0}\frac{e^h-1}{h}$$
$$= e^x\cdot 1$$
$$= e^x$$

The derivative of e^x is itself.

$$\frac{d}{dx}(e^x) = e^x$$

Derivative of e^u

We use the chain rule to find

$$\frac{d}{dx}e^u = e^u\frac{du}{dx}$$

Example:

$$\frac{d}{dx}e^{x+x^2} = e^{x+x^2}\cdot\frac{d}{dx}(x+x^2)$$
$$= e^{x+x^2}(1+2x)$$

Exercise 1

Find

$$\frac{d}{dx}e^{(x^2)}$$

Derivative of a^x

$a > 0$ and $a \neq 1$

Find the derivative of a^x

$$a = e^{\ln a}$$

$$a^x = e^{x \ln a}$$

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{x \ln a}$$

$$= e^{x \ln a} \cdot \frac{d}{dx}(x \ln a)$$

$$= e^{x \ln a} \cdot \ln a = a^x \ln a$$

Example 2

Find the derivative of $y = 8^x$

$$\frac{dy}{dx} = 8^x \ln 8$$

Find the derivative of $y = 3^{\cot x}$

$$\frac{dy}{dx} = 3^{\cot x} \ln 3 \cdot \frac{d}{dx}(\cot x)$$

$$= \ln 3 \cdot 3^{\cot x}(-\csc^2 x)$$

$$= -\ln 3 \cdot 3^{\cot x} \csc^2 x$$

Exercise 2

Find the derivative of $y = 5^{\csc x}$

Reviewing Logarithms

At what point on the graph of $y = 2^x - 3$ does the tangent line have slope 21?

The slope is the derivative:

$$\frac{d}{dx}(2^x - 3) = 2^x \cdot \ln 2 - 0 = 2^x \ln 2$$

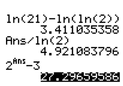
$$2^x \ln 2 = 21$$

$$2^x = \frac{21}{\ln 2}$$

$$\ln 2^x = \ln\left(\frac{21}{\ln 2}\right)$$

$$x \ln 2 = \ln 21 - \ln(\ln 2)$$

$$x = \frac{\ln 21 - \ln(\ln 2)}{\ln 2}$$



The point is approximately $(4.9, 27.3)$

Exercise 3

At what point on the graph of $y = 3^x + 1$ is the tangent line parallel to the line $y = 5x - 1$?

Suppose a glass of cold milk is left on the counter. Its temperature at t minutes is given by

$$T = 72 - 30(0.98^t)$$

1. What is the temperature of the refrigerator?
2. What is the temperature of the room?
3. When is the milk warming fastest?
4. When will the milk reach 55°?
5. At what rate is the milk warming when its temperature is 55°?

Derivative of $\ln x$

$$y = \ln x$$
$$e^y = x$$
$$e^y \frac{dy}{dx} = \frac{dx}{dx}$$
$$e^y \frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

Derivative of $\ln u$

$$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}$$

Example:

$$\frac{d}{dx}\ln\left(\frac{1}{x}\right) = \frac{1}{1/x}\cdot\frac{d}{dx}\left(\frac{1}{x}\right)$$

$$= x\left(-\frac{1}{x^2}\right) = -\frac{1}{x}$$

Exercise 4

Find

$$\frac{d}{dx}(x\ln x - x)$$

Example 5

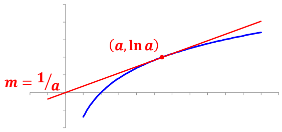
A line with slope m passes through the origin and is tangent to the graph of $y = \ln x$. What is the value of m?

$$m = \frac{\ln a - 0}{a - 0} = \frac{\ln a}{a}$$

$$\frac{\ln a}{a} = \frac{1}{a}$$

$$\ln a = 1$$

$$a = e$$



Exercise 5

A line with slope m passes through the origin and is tangent to $y = \ln(2x)$. What is the value of m?

Derivative of $\log_a x$

$$\log_a x = \frac{\ln x}{\ln a}$$

$$\begin{aligned} \frac{d}{dx}(\log_a x) &= \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right) \\ &= \frac{1}{\ln a} \cdot \frac{d}{dx}(\ln x) \\ &= \frac{1}{\ln a} \cdot \frac{1}{x} \\ &= \frac{1}{x \ln a} \end{aligned}$$

Derivative of $\log_a u$

$a > 0$ and $a \neq 1$

$$\frac{d}{dx}\log_a u = \frac{1}{u \ln a}\frac{du}{dx}$$

Example 6

If $y = \log_a a^{\sin x}$ find dy/dx.

$$\frac{d}{dx}\left(\log_a a^{\sin x}\right) = \frac{1}{a^{\sin x}\ln a} \cdot \frac{d}{dx}\left(a^{\sin x}\right)$$
$$= \frac{1}{a^{\sin x}\ln a} \cdot a^{\sin x}\ln a \cdot \frac{d}{dx}(\sin x)$$
$$= \frac{a^{\sin x}\ln a}{a^{\sin x}\ln a} \cdot \cos x$$
$$= \cos x$$

Exercise 6

If $y = \log_2(1/x)$ find dy/dx.

Power Rule

$$x > 0 \Rightarrow x^n = e^{n\ln x}$$
$$\frac{d}{dx}(x^n) = \frac{d}{dx}\left(e^{n\ln x}\right)$$
$$= e^{n\ln x}\frac{d}{dx}(n\ln x)$$
$$= e^{n\ln x} \cdot \frac{n}{x}$$
$$= x^n \cdot \frac{n}{x} = nx^{n-1}$$

Power Rule for Arbitrary Real Powers

If u is a positive differentiable function of x and n is any real number, then u^n is a differentiable function of x, and

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}.$$

Example 7

a. If $y = x^{\sqrt{2}}$ then

$$\frac{dy}{dx} = \sqrt{2}x^{\sqrt{2}-1}$$

b. If $y = (2 + \sin 3x)^{\pi}$, then

$$\frac{d}{dx}(2 + \sin 3x)^{\pi} = \pi(2 + \sin 3x)^{\pi-1}\cos 3x \cdot 3$$
$$= 3\pi(2 + \sin 3x)^{\pi-1}\cos 3x$$

Exercise 7

If $y = x^{-\sqrt{2}}$ find dy/dx.

Example 8

If $f(x) = \ln(x - 3)$, find $f'(x)$, and state the domain of f'.

The domain of f is $(3, \infty)$, and

$$f'(x) = \frac{1}{x - 3}$$

One is tempted to say that the domain of f' is all real numbers except $x \neq 3$.

However, f is not defined for $x < 3$; therefore, neither is f'.

$$f'(x) = \frac{1}{x - 3}, \qquad x > 3$$

The domain of f' is $(3, \infty)$.

Exercise 8

If $f(x) = \ln(x + 2)$, find $f'(x)$, and state the domain of f'.

Example 9
Logarithmic Differentiation

If $y = x^x$ find dy/dx.

We can find the derivative by first taking the log of both sides.

$$\ln y = \ln x^x$$

$$\ln y = x \ln x$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(x \ln x)$$

$$\frac{1}{y}\frac{dy}{dx} = 1 \cdot \ln x + x \cdot \frac{1}{x}$$

$$\frac{1}{y}\frac{dy}{dx} = \ln x + 1$$

$$\frac{dy}{dx} = y(\ln x + 1)$$

$$\frac{dy}{dx} = x^x(\ln x + 1)$$

Exercise 9

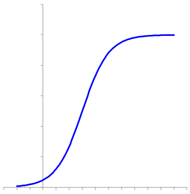
Use the technique of logarithmic differentiation to find

$$\frac{d}{dx}\left((\sin x)^x\right), \qquad 0 < x < \frac{\pi}{2}$$

Example 10

The spread of the flu in a certain school is modeled by

$$P(t) = \frac{100}{1 + e^{3-t}}$$



a. Estimate the initial number of students infected.

$$P(0) = \frac{100}{1 + e^{3-0}} = 5$$

b. How fast is the flu spreading after 3 days?

$$\frac{dP}{dt} = \frac{d}{dt}(100(1 + e^{3-t})^{-1})$$

$$= 100 \cdot (-1) \cdot (1 + e^{3-t})^{-2} \cdot \frac{d}{dt}(1 + e^{3-t})$$

$$= -100(1 + e^{3-t})^{-2} \cdot (0 + e^{3-t}) \frac{d}{dt}(3 - t)$$

$$= -100(1 + e^{3-t})^{-2} \cdot e^{3-t}\,(-1)$$

$$= \frac{100e^{3-t}}{(1 + e^{3-t})^2}$$

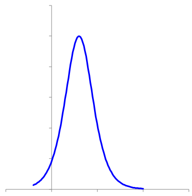
$$t = 3 \Rightarrow P(3) = \frac{100 \cdot 1}{(1 + 1)^2} = 25$$

The flu is spreading to 25 new students per day.

c. When will the flu spread at its maximum rate, and what is this rate?

To find the maximum rate, we should find the time when the derivative is a maximum.

Looking at the graph of the derivative, the peak appears to occur at 3 days and has a value of 25 students per day.



Exercise 10

During the early 1960's, several nations tested nuclear weapons in the atmosphere. The fallout from these tests included significant amounts of radioactive ^{90}Sr. Chemically, strontium is very similar to calcium, and it can be incorporated into growing bones and teeth.

The half life of ^{90}Sr is 28.8 years. The amount of ^{90}Sr remaining after t years is $A = A_0(1/2)^t$. Find the decay rate after 10 years.

Homework

p 178: 3-48 multiples of 3, 53, 54, 56, 64