Chapter 3.8

Derivatives of Inverse Trigonometric Functions

Objectives

- Calculate derivatives of functions Involving the Inverse Trigonometric Functions.

Learning Target

- 80% of the students will be able to calculate the derivative of $\cos^{-1} x$.

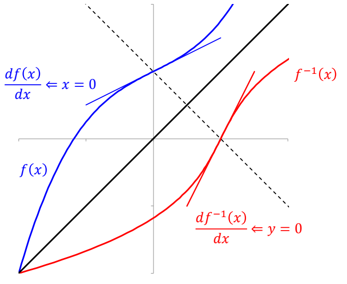
Standard

G-C.4 Construct a tangent line from a point outside a given circle to the circle.

Overview

- Derivatives of Inverse Functions
- Derivative of the Arcsine
- Derivative of the Arctangent
- Derivative of the Arcsecant
- Derivatives of the Other Three

Inverse Functions

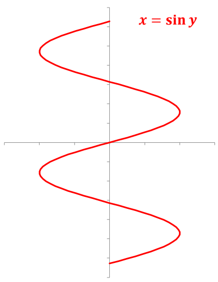


Derivatives of Inverse Functions

If f is differentiable at every point of an interval I and df/dx is never zero on I, then f has an inverse and f^{-1} is differentiable at every point of the interval $f(I)$.

Derivative of the Arcsine

- $x = \sin y$ differentiable in $-\pi/2 < y < \pi/2$
- $\cos y > 0$
- $y = \sin^{-1} x$ differentiable in $-1 < x < 1$
- Differentiable at $x = \pm 1$?



$$y = \sin^{-1} x$$

$$\sin y = x$$

$$\frac{d}{dx}(\sin y) = \frac{d}{dx}x$$

$$\cos y \frac{dy}{dx} = 1$$

$$-\frac{\pi}{2} < y < \frac{\pi}{2} \Rightarrow \cos y > 0$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

$$\cos y = \sqrt{1 - (\sin y)^2} = \sqrt{1 - x^2}$$

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}$$

u is differentiable function of x, $|u| < 1$ then

$$\frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1 - u^2}} \frac{du}{dx}$$

Example 1

Find the derivative of $\sin^{-1}(x^2)$

Let $u = x^2$

$$\frac{d}{dx}(\sin^{-1}(x^2)) = \frac{d}{dx}(\sin^{-1} u) \frac{du}{dx}$$

$$= \frac{1}{\sqrt{1 - u^2}} \frac{d}{dx}(x^2)$$

$$= \frac{2x}{\sqrt{1 - x^4}}$$

Exercise 1

Find

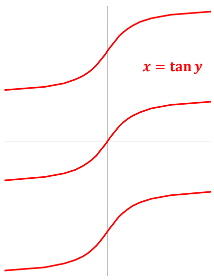
$$\frac{d}{dx}(\sin^{-1}(1 - x)) \qquad 0 < x < 1$$

Derivative of Arctangent

Domain of $y = \tan^{-1} x$ is $\mathbb{R}$.

Differentiable in $\mathbb{R}$.

$$y = \tan^{-1} x$$
$$\tan y = x$$
$$\frac{d}{dx}(\tan y) = \frac{d}{dx} x$$
$$\sec^2 y \frac{dy}{dx} = 1$$



$$\frac{dy}{dx} = \frac{1}{\sec^2 y}$$
$$\sec^2 y = 1 + \tan^2 y$$
$$\frac{dy}{dx} = \frac{1}{1 + \tan^2 y}$$
$$\frac{dy}{dx} = \frac{1}{1 + x^2}$$

If u is differentiable function of x then

$$\frac{d}{dx} \tan^{-1} u = \frac{1}{1 + u^2} \frac{du}{dx}$$

Example 2

A particle moves along the x-axis: $t \geq 0 \Rightarrow$ $x(t) = \tan^{-1} \sqrt{t}$.

What is the velocity when $t = 16$?

$$v(t) = \frac{d}{dt} \tan^{-1} \sqrt{t}$$
$$= \frac{1}{1 + \left(\sqrt{t}\right)^2} \cdot \frac{d}{dt} \sqrt{t} = \frac{1}{1 + t} \cdot \frac{1}{2\sqrt{t}}$$
$$v(16) = \frac{1}{1 + 16} \cdot \frac{1}{2\sqrt{16}} = \frac{1}{136}$$

Exercise 2

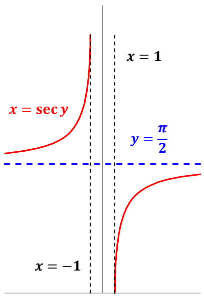
A particle moves along the x-axis: $t \geq 0 \Rightarrow$
$x(t) = \tan^{-1} t$.

What is the velocity when $t = 2$?

Derivative of Arcsecant

Find the derivative of
$y = \sec^{-1} x$ when $|x| > 1$

$$y = \sec^{-1} x$$
$$\sec y = x$$
$$\frac{d}{dx}(\sec y) = \frac{d}{dx} x$$
$$\sec y \tan y \frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{1}{\sec y \tan y}$$



$$\sec y = x$$
$$\tan y = \pm\sqrt{\sec^2 y - 1}$$
$$\frac{dy}{dx} = \pm\frac{1}{x\sqrt{x^2 - 1}}$$
$$\frac{dy}{dx} > 0$$
$$\frac{dy}{dx} = \frac{1}{|x|\sqrt{x^2 - 1}}$$

Example 3

Find

$$\frac{d}{dx}\sec^{-1}(5x^4)$$

$$= \frac{1}{|5x^4|\sqrt{(5x^4)^2 - 1}}\frac{d}{dx}(5x^4)$$

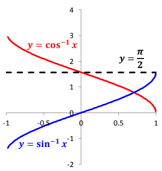
$$= \frac{20x^3}{5x^4\sqrt{25x^8 - 1}} = \frac{4}{x\sqrt{25x^8 - 1}}$$

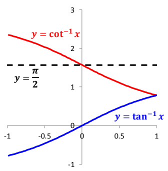
Exercise 3

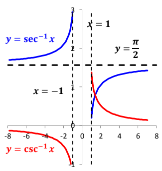
Find

$$\frac{d}{dt}\sec^{-1}\frac{1}{t}, \qquad 0 < t < 1$$

Other Inverse Functions







$\cos^{-1}x = \frac{\pi}{2} - \sin^{-1}x$ $\qquad$ $\cot^{-1}x = \frac{\pi}{2} - \tan^{-1}x$ $\qquad$ $\csc^{-1}x = \frac{\pi}{2} - \sec^{-1}x$

$\frac{d}{dx}\cos^{-1}x = -\frac{d}{dx}\sin^{-1}x$ $\qquad$ $\frac{d}{dx}\cot^{-1}x = -\frac{d}{dx}\tan^{-1}x$ $\qquad$ $\frac{d}{dx}\csc^{-1}x = -\frac{d}{dx}\sec^{-1}x$

Example 4

Find the equation for the line tangent to the graph of $y = \cot^{-1} x$ at $x = -1$.

$$\cot^{-1}(-1) = \frac{\pi}{2} - \tan^{-1}(-1) = \frac{\pi}{2} - \left(-\frac{\pi}{4}\right) = \frac{3\pi}{4}$$

$$\left.\frac{d \cot^{-1} x}{dx}\right]_{x=-1} = -\left.\frac{1}{1+x^2}\right]_{x=-1} = -\frac{1}{2}$$

$$y - \frac{3\pi}{4} = -\frac{1}{2}(x+1)$$

Exercise 4

Find an equation for the line tangent to the graph of $y = \sec^{-1} x$ at $x = 2$.

Homework

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