Chapter 3.7

Implicit Differentiation

Objectives

- Find derivatives using implicit differentiation.
- Find derivatives using the Power Rule for Rational Powers of x.

Learning Target

- 80% of the students will be able to calculate the derivative of $\sqrt[3]{x}$.

Standard

G-C.4 Construct a tangent line from a point outside a given circle to the circle.

Overview

- Implicitly Defined Functions
- Lenses, Tangents and Normal Lines
- Derivatives of Higher Order
- Rational Powers of Differentiable Functions

Implicitly Defined Functions

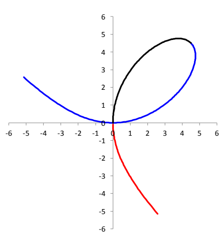
Folium

Descartes (1638)

$x^3 + y^3 - 9xy = 0$

Solve for y?

What is y'?



Implicit Differentiation

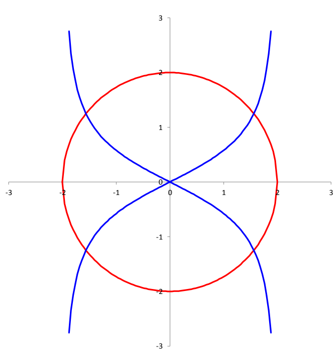
- When a function is written in the form $y = f(x)$, y is called an explicit function of x.
- An equation like $x^2 + y^2 = 4$ can involve y as an implicit function of x.
- When we differentiate the equation with respect to x, we get:
- $\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(4)$
- $2x + 2y \cdot \frac{dy}{dx} = 0$

Implicit Differentiation

- $\frac{d}{dx}(y^2)$ uses the Chain Rule since y is a function of x.
- Solving for $\frac{dy}{dx}$ we get $2y \cdot \frac{dy}{dx} = -2x \Rightarrow$
- $\frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}$.
- Verify the answer graphically.

$x^2 + y^2 = 4$

$\frac{dy}{dx} = -\frac{x}{y}$



Exercise 1

Find y' for Descartes' folium:

$$x^3 + y^3 - 9xy = 0$$

Slope of a Circle

When we found the slope of the circle, we found,

$$\frac{dy}{dx} = -\frac{x}{y}$$

This single equation gives us the slope for both the upper half and the lower half of the circle.

Example 2

Show that dy/dx is defined at every point on the graph of $2y = x^2 + \sin y$.

Find dy/dx by using implicit differentiation.

$$2y = x^2 + \sin y$$

$$\frac{d}{dx}(2y) = \frac{d}{dx}(x^2 + \sin y)$$

$$2\frac{dy}{dx} = 2x + \cos y\frac{dy}{dx}$$

$$2\frac{dy}{dx} - \cos y\frac{dy}{dx} = 2x$$

$$(2 - \cos y)\frac{dy}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{2x}{2 - \cos y}$$

Since $\cos y \neq 2$, dy/dx is defined at every point (x, y)

Exercise 2

Find where the slope of the following curve is defined:

$$x^2y - xy^2 = 4$$

Implicit Differentiation

1. Differentiate both sides of the equation with respect to x.
2. Collect the terms with dy/dx on one side of the equation.
3. Factor out dy/dx.
4. Solve for dy/dx.

Snell's Law

When light passes through an interface between two regions with different indices of refraction, the light's path is described in terms of the angle between its path and the line normal to the surface.

The surface of a lens is often defined by a quadratic function.

If this is the case, we can use implicit differentiation to find the equation of the normal.

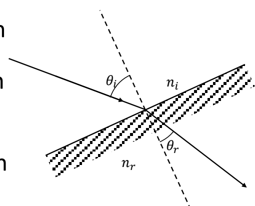
Snell's Law

θ_i is the angle of incidence

θ_r is the angle of refraction

n_i is the index of refraction on the incident side

n_r is the index of refraction on the refracted side



$$n_i \sin \theta_i = n_r \sin \theta_r$$

Example 3
Tangent and Normal to an Ellipse

Find the tangent and the normal to the ellipse $x^2 - xy + y^2 = 7$ at the point $(-1, 2)$.

$$x^2 - xy + y^2 = 7$$

$$\frac{d}{dx}(x^2) - \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = \frac{d}{dx}(7)$$

$$2x - \left(x\frac{dy}{dx} + y\frac{dx}{dx}\right) + 2y\frac{dy}{dx} = 0$$

$$2x - y - x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$
$$(2y - x)\frac{dy}{dx} = y - 2x$$
$$\frac{dy}{dx} = \frac{y - 2x}{2y - x}$$

If $x = -1$ and $y = 2$ then,

$$\frac{dy}{dx} = \frac{2 - 2(-1)}{2 \cdot 2 - (-1)} = \frac{4}{5}$$

The tangent at $(-1, 2)$ is,

$$y - 2 = \frac{4}{5}\big(x - (-1)\big)$$
$$y = \frac{4}{5}x + \frac{14}{5}$$

The normal at $(-1, 2)$ is,

$$y - 2 = -\frac{5}{4}\big(x - (-1)\big)$$
$$y = -\frac{5}{4}x + \frac{3}{4}$$

Exercise 3

Find the lines that are tangent and normal to $x^2 + xy - y^2 = 1$ at $(2, 3)$.

Example 4
Finding a Second Derivative Implicitly

Find d^2y/dx^2 if $2x^3 - 3y^2 = 8$.

$$\frac{d}{dx}(2x^3 - 3y^2) = \frac{d}{dx}(8)$$

$$6x^2 - 6yy' = 0$$

$$x^2 - yy' = 0$$

$$y' = \frac{x^2}{y} \qquad y \neq 0$$

Now, find the second derivative,

$$y'' = \frac{d}{dx}\left(\frac{x^2}{y}\right) = \frac{2xy - x^2y'}{y^2} = \frac{2x}{y} - \frac{x^2}{y^2} \cdot y'$$

Substitute $y' = x^2/y$.

$$y'' = \frac{2x}{y} - \frac{x^2}{y^2} \cdot \frac{x^2}{y} = \frac{2x}{y} - \frac{x^4}{y^3}, \qquad y \neq 0$$

Exercise 4

Use implicit differentiation to find dy/dx and then d^2y/dx^2 if $y^2 = x^2 + 2x$.

Exploration

What does the graph of $x^2 - 2xy + y^2 = 4$ look like?

1. Use implicit differentiation to find dy/dx.
2. What do you conjecture about the graph?
3. What are the possible values of y when $x = 0$?

The original equation can be written as $(x - y)^2 - 4 = 0$

4. Factor the expression on the left and write two equations whose graphs combine to give the graph of the original equation.
5. Sketch the graph.
6. Explain why your graph is consistent with the derivative from part 1.

Derivatives of Rational Powers

Let p and q be integers with $q > 0$. Moreover,

$y = \sqrt[q]{x^p} = x^{p/q}$.

$$y^q = x^p$$

Differentiate both sides of the equation with respect to x.

$$qy^{q-1}\frac{dy}{dx} = px^{p-1}$$

If $y \neq 0$ then,

$$\begin{aligned}\frac{dy}{dx} &= \frac{px^{p-1}}{qy^{q-1}}\\ &= \frac{p}{q} \cdot \frac{x^{p-1}}{(x^{p/q})^{q-1}}\\ &= \frac{p}{q} \cdot \frac{x^{p-1}}{x^{p-p/q}}\\ &= \frac{p}{q} \cdot x^{(p-1)-(p-p/q)}\\ &= \frac{p}{q} \cdot x^{(p/q)-1}\end{aligned}$$

Power Rule for Rational Powers of x

If n is any rational number, then

$$\frac{d}{dx}x^n = nx^{n-1}.$$

If $n < 1$. then the derivative does not exist at $x = 0$.

Combining the Chain Rule with the Power Rule

If n is a rational number and u is a differentiable function of x, then u^n is a differentiable function of x:

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}$$

Provided $u \neq 0$ if $n = 1$.

Example 5

$$\frac{d}{dx}\sqrt{x} = \frac{d}{dx}\left(x^{1/2}\right) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

$\sqrt{x}$ is defined at $x = 0$, but $1/(2\sqrt{x})$ is not.

$$\frac{d}{dx}\left(x^{2/3}\right) = \frac{2}{3}\left(x^{-1/3}\right) = \frac{2}{3x^{1/3}} = \frac{2}{3\sqrt[3]{x}}$$

$x^{2/3}$ is defined at $x = 0$, but $2/(3\sqrt[3]{x})$ is not.

$$\frac{d}{dx}(\cos x)^{-1/5} = -\frac{1}{5}(\cos x)^{-6/5} \cdot \frac{d}{dx}(\cos x)$$

$$= -\frac{1}{5}(\cos x)^{-6/5} \cdot (-\sin x)$$

$$= \frac{1}{5}\sin x\,(\cos x)^{-6/5}\,.$$

Exercise 5

Find dy/dx if $y = \sqrt[4]{x}$.

Homework

p 162: 3-53 multiples of 3, 54, 58