# Chapter 3.6

Chain Rule

## Objectives

- Differentiate composite functions using the Chain Rule.
- Find the slopes of parameterized curves.

## Learning Target

- 80% of the students will be able to calculate the derivative of $\csc(x^2)$.

## Standard

G-C.4 Construct a tangent line from a point outside a given circle to the circle.

## Overview

- Derivative of a Composite Function
- “Outside –Inside” Rule
- Repeated use of the Chain Rule
- Slopes of Parameterized Curve $s$
- Power Chain Rule

## Derivatives of Composite Functions

$y = 6x - 10 = 2(3x - 5)$

$y = 2u \qquad u = 3x - 5$

$\frac{dy}{dx} = 6 \qquad \frac{dy}{du} = 2 \qquad \frac{du}{dx} = 3$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

## Derivatives of Composite Functions

$y = 9x^4 + 6x^2 + 1 = (3x^2 + 1)^2$

$y = u^2 \qquad u = 3x^2 + 1$

$\frac{dy}{du} = 2u = 6x^2 + 2$

$\frac{du}{dx} = 6x$

$\frac{dy}{du} \cdot \frac{du}{dx} = (6x^2 + 2) \cdot 6x = 36x^3 + 12x$

$\frac{dy}{dx} = 36x^3 + 12x$

## Chain Rule

$$\frac{d}{dx}\left(f\left(g(x)\right)\right) = f'\left(g(x)\right) \cdot g'(x)$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

## Example 1

An object moves along the $x$-axis so that its position at any time $t \geq 0$ is

$$x(t) = \cos(t^2 + 1)$$

Find its velocity as a function of $t$.

$$x(u) = \cos u$$

$$u = t^2 + 1$$

## Example 1 continued

$$\frac{dx}{du} = -\sin u$$
$$\frac{du}{dt} = 2t$$
$$\frac{dx}{dt} = \frac{dx}{du} \cdot \frac{du}{dt}$$
$$= -\sin u \cdot 2t$$
$$= -\sin(t^2+1) \cdot 2t \ = -2t \sin(t^2+1)$$

## Exercise 1

An object moves along the $x$-axis so that its position at any time $t \geq 0$ is

$$x(t) = t\cos(\pi - 4t)$$

Find its velocity as a function of $t$.

## Example 2
### "Outside-Inside" Rule

1. Differentiate the "outside" function.
2. Multiply by the derivative of the inside function.

$$y = (2x^2 - 3x + 1)^2$$

Outside function: $\bigcirc^2$

Inside Function: $\bigcirc = 2x^2 - 3x + 1$

Derivative: $2\bigcirc \cdot \frac{d}{dx}\bigcirc$

$$= 2(2x^2 - 3x + 1) \cdot (4x - 3)$$

## “Outside-Inside” Rule

$$y = \sin(x^2 + x)$$

Outside function: $\sin$ 

Inside Function:  $= x^2 + x$

Derivative: $y' = \cos$  $\cdot \frac{d}{dx}$ 

$$= \cos(x^2 + x) \cdot (2x + 1)$$

## Exercise 2

$y = (\csc x + \cot x)^{-1}$

Find $y'$

## Example 3
### Repeating the Chain Rule

Use the Chain Rule two or more times.

Find the derivative of $g(t) = \tan(5 - \sin 2t)$.

$$g'(t) = \frac{d}{dt}(\tan(5 - \sin 2t))$$
$$= \sec^2(5 - \sin 2t) \cdot \frac{d}{dt}(5 - \sin 2t)$$
$$= \sec^2(5 - \sin 2t) \cdot \left(0 - \cos 2t \frac{d}{dt}(2t)\right)$$
$$= \sec^2(5 - \sin 2t) \cdot (-2\cos 2t)$$
$$= -2\cos(2t)\sec^2(5 - \sin 2t)$$

## Exercise 3

Find

$$\frac{d}{dx}\left((1+\cos^2(7x))^3\right)$$

## Parameterized Curves

A parameterized curve $\left(x(t), y(t)\right)$ is *differentiable at t* if $x$ and $y$ are differentiable at $t$. At a point on the differentiable curve where $y$ is also a differentiable function of $x$, the derivative of $y$ is,

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

**Finding $dy/dx$ Parametrically**

If all three derivatives exist and $dx/dt \neq 0$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

## Example 4
### Differentiating with a Parameter

$x = \sec t$

$y = \tan t$

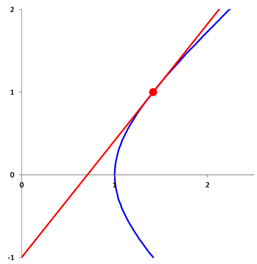
$-\frac{\pi}{2} < t < \frac{\pi}{2}$

$\frac{dx}{dt} = \sec t \tan t$

$\frac{dy}{dt} = \sec^2 t$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sec^2 t}{\sec t \tan t}$

$\frac{dy}{dx} = \frac{\sec t}{\tan t} = \csc t$



## Exercise 4

$x = 2\cos t \qquad y = 2\sin t$

Find the equation of the line tangent to the curve at $t = \pi/4$.

## Example 5
### Power Chain Rule

$y = \bigcirc^n, \quad \frac{dy}{dx} = n\bigcirc^{n-1} \cdot \frac{d}{dx}\bigcirc$

$y = (x^5 - 7x)^9$ find $y'$

$$y' = 9(x^5 - 7x)^8 \cdot \frac{d}{dx}(x^5 - 7x)$$

$$y' = 9(x^5 - 7x)^8 \cdot (5x^4 - 7)$$

## Exercise 5

$y = (2x^2 - 3x + 1)^3$
Find $y'$

## Example 6
### Finding Slope

A. Find the slope of the line tangent to $y = \sin^5 x$ at the point where $x = \pi/3$.

$$\frac{dy}{dx} = 5\sin^4 x \cdot \frac{d}{dx}\sin x$$
$$= 5\sin^4 x \cos x$$
$$x = \pi/3 \Rightarrow \frac{dy}{dx} = 5\left(\frac{\sqrt{3}}{2}\right)^4\left(\frac{1}{2}\right) = \frac{45}{32}$$

B. Show that the slope of every line tangent to the curve $y = 1/(1-2x)^3$ is positive.

$$\frac{dy}{dx} = \frac{d}{dx}\left((1-2x)^{-3}\right)$$
$$= -3(1-2x)^{-4} \cdot \frac{d}{dx}(1-2x)$$
$$= -3(1-2x)^{-4} \cdot (-2)$$
$$= \frac{6}{(1-2x)^4} > 0 \quad \Leftarrow \quad x \neq \frac{1}{2}$$

## Exercise 6

What is the largest possible value for the slope of the curve $y = \sin(x/2)$?

## Example 7
### Radians vs Degrees

$$\frac{d}{dx}\sin(x^\circ) = \frac{d}{dx}\sin\left(\frac{\pi}{180}x\right)$$
$$= \frac{\pi}{180}\cos\left(\frac{\pi}{180}x\right) = \frac{\pi}{180}\cos(x^\circ)$$

$$\frac{d}{dx}\cos(x^\circ) = \frac{d}{dx}\cos\left(\frac{\pi}{180}x\right)$$
$$= \frac{\pi}{180}\left(-\sin\left(\frac{\pi}{180}x\right)\right) = -\frac{\pi}{180}\sin(x^\circ)$$

## Homework

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