Chapter 3.5

Derivatives of Trigonometric Functions

Objectives

- Use the rules for differentiating the six basic trigonometric functions.

Learning Target

- 80% of the students will be able to calculate the derivative of $\csc x$.

Standard

G-C.4 Construct a tangent line from a point outside a given circle to the circle.

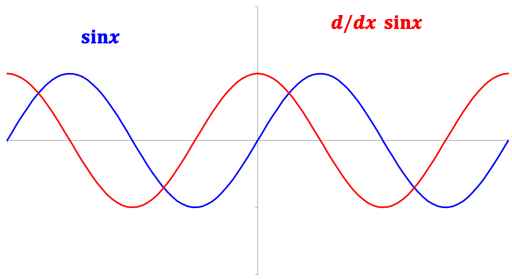
Overview

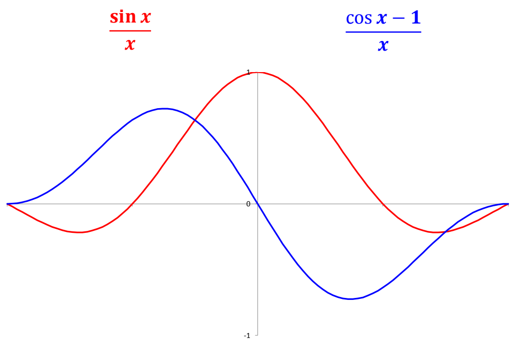
- Derivative of the Sine Function
- Derivative of the Cosine Function
- Simple Harmonic Motion
- Jerk
- Derivatives of Other Basic Trigonometric Functions

Periodic Motion

- Many phenomena are periodic.
  - Sound
  - Light
  - Ocean waves
    - Tides
    - Tsunamis
  - Earthquakes
  - Heart rates

Derivative of $\sin x$





Derivative of $\sin x$

Let $y = \sin x$

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x\,(\cos h - 1) + \cos x \sin h}{h}$$

$$= \lim_{h\to 0} \sin x \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} + \lim_{h\to 0} \cos x \cdot \lim_{h\to 0} \frac{\sin h}{h}$$

$$= \sin x \cdot 0 + \cos x \cdot 1$$

$$= \cos x$$

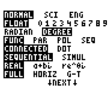
Derivative of $\sin x$ and $\cos x$

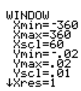
$$\frac{d}{dx}\sin x = \cos x$$

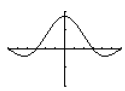
$$\frac{d}{dx}\cos x = -\sin x$$

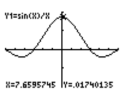
The sine and cosine functions are differentiable.
Continuous over their domains.
Also true of the other basic trigonometric functions

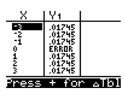
Using Degrees











Example 1

a) Find the derivative of $y = x^2 \sin x$.

$$\frac{dy}{dx} = x^2 \cdot \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(x^2)$$

$$= x^2 \cos x + 2x \sin x$$

b) Find the derivative of $u = \frac{\cos x}{1-\sin x}$

$$\frac{du}{dx} = \frac{(1-\sin x)\frac{d}{dx}(\cos x) - \cos x \frac{d}{dx}(1-\sin x)}{(1-\sin x)^2}$$

$$= \frac{(1-\sin x)(-\sin x) - \cos x\,(0-\cos x)}{(1-\sin x)^2}$$

$$= \frac{-\sin x + \sin^2 x + \cos^2 x}{(1-\sin x)^2}$$

$$= \frac{1-\sin x}{(1-\sin x)^2} = \frac{1}{1-\sin x}$$

Exercise 1

Find the derivative of

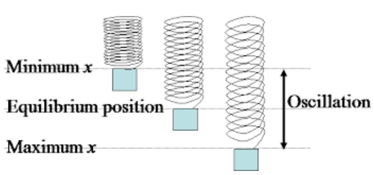
a) $y = 3x + x\tan x$

b) $y = \frac{\cos x}{1+\sin x}$

Example 2
Simple Harmonic Motion

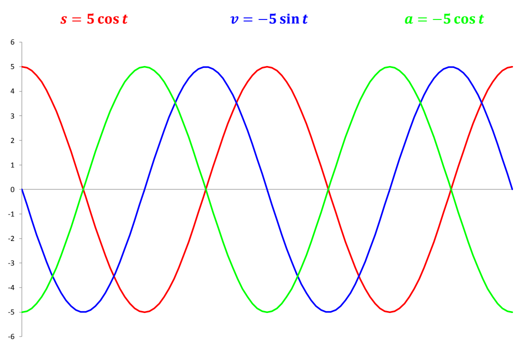
A weight hanging from a spring is stretched 5 units below its rest position ($s = 0$) and released at $t = 0$ to bob up and down. Its position at any time t is $s = 5\cos t$.

What are its velocity and acceleration at t?



Solution

- Position
  $s = 5\cos t$
- Velocity
  $$v = \frac{d}{dt}(5\cos t) = -5\sin t$$
- Acceleration
  $$a = \frac{dv}{dt} = \frac{d}{dt}(-5\sin t) = -5\cos t$$



Exercise 2

A body is moving in simple harmonic motion with position $s = \cos t - 3\sin t$.

1. Find its velocity, speed, and acceleration at time t.

2. Find its velocity, speed, and acceleration at time $t = \pi/4$.

Jerk

- Not the boyfriend who dumped you.
- Sudden change in acceleration.
- The 3rd derivative of position.

Definition
Jerk is the derivative of acceleration. If a body's position at time t is given by $s(t)$,

$$j(t) = \frac{da}{dt} = \frac{d^3s}{dt^3}.$$

Example 3

1. The jerk caused by the acceleration of gravity ($g = 9.8\ m/s^2$) is zero.

$$j = \frac{d}{dt}g = 0$$

We do not experience motion sickness while sitting in a chair.

Example 3

2. The jerk of the simple harmonic motion of Example 2 is,

$$j = \frac{da}{dt} = \frac{d}{dt}(-5\cos t)$$
$$= 5\sin t$$

Greatest magnitude when body at equilibrium position ($\sin t = \pm 1$) where acceleration is zero.

Exercise 3

A body is moving in simple harmonic motion with its position given by,

$$s(t) = 2 + 2\sin t$$

Find the jerk at time t.

Other Trigonometric Functions

$$\tan x = \frac{\sin x}{\cos x} \qquad \frac{d}{dx}\tan x = \sec^2 x$$

$$\csc x = \frac{1}{\sin x} \qquad \frac{d}{dx}\csc x = -\csc x \cot x$$

$$\sec x = \frac{1}{\cos x} \qquad \frac{d}{dx}\sec x = \sec x \tan x$$

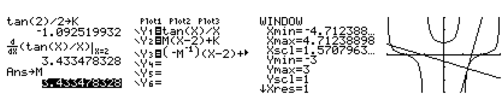
$$\cot x = \frac{\cos x}{\sin x} \qquad \frac{d}{dx}\cot x = -\csc^2 x$$

Example 4

Find the equations of the lines that are tangent and normal to

$$f(x) = \frac{\tan x}{x}$$

at $x = 2$. support graphically.



Exercise 4

Find the equations of the lines that are tangent and normal to the graph of $y = \sec x$ at $x = \pi/4$.

Example 5

Find y'' if $y = \sec x$.

$$y = \sec x$$
$$y' = \sec x \tan x$$
$$y'' = \frac{d}{dx}(\sec x \tan x)$$
$$= \sec x \frac{d}{dx}(\tan x) + \tan x \frac{d}{dx}(\sec x)$$
$$= \sec x(\sec^2 x) + \tan x\,(\sec x \tan x)$$
$$= \sec^3 x + \sec x \tan^2 x$$

Exercise 5

Find y'' if $y = \theta \tan \theta$.

Homework

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