Chapter 3.4

Velocity and Other Rates of Change

Objectives

- Use derivatives to analyze straight line motion.
- Solve problems involving rates of change.

Learning Target

- 80% of the students will be able to graph the trajectory of an object tossed from a height of 2 m at an angle of 45° relative to the Earth's surface with an initial velocity of 60 m/s.

Standard

G-C.4 Construct a tangent line from a point outside a given circle to the circle.

Overview

- Instantaneous Rate of Change
- Motion along a Line
- Sensitivity to Change
- Derivatives in Economics

Instantaneous Rate of Change

The (instantaneous) rate of change of f at a is the derivative

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided the limit exists.

Use "instantaneous" even when x is not time.

Often omit "instantaneous".

Say "rate of change".

Example 1
Enlarging Circles

a) Find the rate of change of the area A of the area of a circle with respect to its radius r.

b) Evaluate the rate of change of A at $r = 5$ and at $r = 10$.

c) If r is measured in inches and A is measured in square inches, what units would be appropriate for dA/dr?

Example 1
Solution

The area of a circle is related to the radius by the equation $A = \pi r^2$.

a) The instantaneous rate of change of A with respect to r is

$$\frac{dA}{dr} = \frac{d}{dr}(\pi r^2) = \pi \cdot 2r = 2\pi r$$

b) At $r = 5$ the rate is 10π. At $r = 10$ the rate is 20π.

Example 1
Solution

The rate of change increases as the radius increases.

c) The appropriate units for dA/dr are square inches (of area) per inch (of radius).

Exercise 1

a) Write the area A of a circle as a function of its circumference C.
b) Find the (instantaneous) rate of change of the area A with respect to the circumference C.
c) Evaluate the rate of change at $C = \pi$ and at $C = 6\pi$.
d) If C is measured in inches and A is measured in square inches, what units would be appropriate for dA/dC?

Tree Rings

Notice that the thickness of the tree rings in this photograph decrease as the radius increases.

What does that tell us about the amount of wood added each successive year?

Motion along a Line

Suppose an object is moving along a coordinate line (say s), and we know its position s as a function of time.

$$s = f(t)$$

The ***displacement*** of the object over the time interval t to $t + \Delta t$ is

$$\Delta s = f(t + \Delta t) - f(t)$$

The ***average velocity*** of the object over that time interval is

$$v_{av} = \frac{displacement}{travel\ time} = \frac{\Delta s}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t}$$

To find the exact (instantaneous) velocity at t, we take the limit as Δt goes to 0.

$$\frac{ds}{dt}$$

Instantaneous Velocity

The instantaneous velocity is the derivative of the position function $s = f(t)$ with respect to time. At time t, the velocity is

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}$$

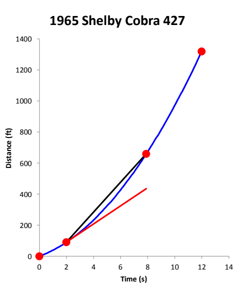
Velocity of a Car

Slope of secant (black) line

$$m = \frac{660 - 91}{7.9 - 2} = 97\ {}^{ft}/_{s}$$

Slope of tangent (red) line is speedometer reading at 2 seconds.

$$m = 59\ {}^{ft}/_{s}$$



Velocity vs Speed

- ***Velocity*** is the rate of change and the direction.
- ***Speed*** is the absolute value of the velocity.
- A horse on a merry-go-round has a constant speed, but its velocity is constantly changing.

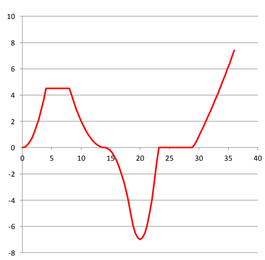
Exercise 2

The number of gallons of water in a tank t minutes after the tank has started to drain is $Q(t) = 200(30 - t)^2$.

a. How fast is the water running out of the tank after 10 minutes?
b. What is the average rate at which the water flows out during the first 10 minutes?

Example 3
Reading a Velocity Graph

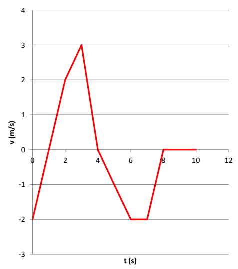
$0 \le t < 4$ forward
$4 \le t < 8$ forward
$8 \le t < 14$ forward
$14 \le t < 23$ backward
$23 \le t < 29$ stationary
$29 \le t < 36$ forward



Exercise 3

Use this graph of velocity vs time to answer the following questions:

a. When does the particle move forward? Backward? Speed up? Slow down?
b. When is the particle's acceleration positive? Negative? Zero?
c. When does the particle move at its greatest speed?
d. When does the particle stand still for more than an instant?



Acceleration

- The derivative of velocity with respect to time.

$$a = \frac{dv}{dt} = \dot{v}$$

- The second derivative of position with respect to time.

$$a = \frac{d^2 s}{dt^2} = \ddot{s}$$

Free Fall

- A body at rest released near the Earth's surface will fall according to

$$s = \frac{1}{2} g t^2$$

$$g = 9.8\ m/s^2$$

$$g = 32\ ft/s^2$$

Example 4
Modeling Vertical Motion

- A dynamite explosion propels a heavy rock straight up with a launch velocity of 160 ft/s.
- It reaches a height of $s = 160t - 16t^2$ after t seconds.
- How high does the rock go?
  - Maximum height when $v = 0$
  - $\frac{ds}{dt} = 160 - 32t = 0$

  - $32t = 160$
  - $t = 5\,s$
  - $s_{max} = s(5) = 160 \cdot 5 - 16 \cdot 5^2 = 400\,ft$
- What is the vertical speed of the rock when it is 256 ft above the ground on the way up? On the way down?
  - $s(t) = 160t - 16t^2 = 256$
  - $16t^2 - 160t + 256 = 0$
  - $t^2 - 10t + 16 = 0$
  - $(t-2)(t-8) = 0$

  - $t = 2, 8\,s$
  - $v = 160 - 32 \cdot 2 = 96\,ft/s$
  - $v = 160 - 32 \cdot 8 = -96\,ft/s$
- What is the rock's acceleration at any time during its flight (after the explosion)?
  - $a(t) = \frac{d^2s}{dt^2} = \frac{d^2}{dt^2}(160t - 16t^2) = -32\ ft/s^2$
- When does the rock hit the ground?
  - $s(t) = 160t - 16t^2 = 0$
  - $10t - t^2 = 2$
  - $t = 10\,s$

Exercise 4

- A rock thrown vertically upward from the surface of the moon with an initial velocity of $24\ m/s$ reaches a height of $s = 24t - 0.8t^2$.

a. Find the rock's velocity and acceleration as functions of time.
b. How long did it take the rock to reach its highest point?
c. How high did the rock go?
d. When did the rock reach half its maximum height?
e. How long was the rock aloft?

Example 5
Studying Particle Motion

A particle moves along a line with position given by

$$s(t) = t^2 - 4t + 3 \quad t \geq 0,$$

where s is in meters, and t is in seconds.

a) Find the displacement of the particle during the first 2 s.

$$\Delta s = s(2) - s(0) = (-1) - 3 = -4$$

b) Find the average velocity of the particle during the first 4 seconds.

$$v_{av} = \frac{s(4) - s(0)}{4 - 0} = \frac{3 - 3}{4} = 0$$

c) Find the instantaneous velocity when $t = 4$.

The velocity is

$$v(t) = \frac{ds}{dt} = 2t - 4$$

$$v(4) = 2 \cdot 4 - 4 = 4\ m/s$$

d) Find the acceleration of the particle when $t = 4$.

$$a(t) = \frac{d^2s}{dt^2} = 2\ m/s^2$$

$$a(4) = 2\ m/s^2$$

e) Describe the motion of the particle. At what values of t does the particle change directions?

Use graphs of s and v to help answer this question.

$0 \le t < 2 \Rightarrow v < 0$

moves left

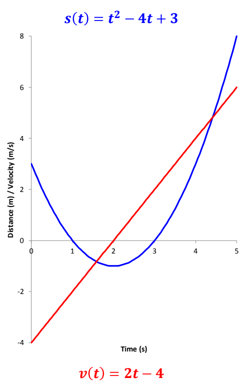
$t = 2 \Rightarrow v = 0$

at rest

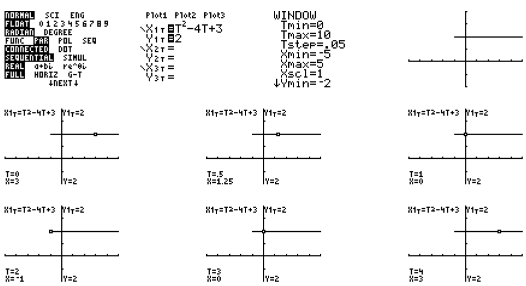
$t > 2 \Rightarrow v > 0$

moves right

Changes direction at $t = 2$



f) Use parametric graphing to view the motion of the particle on the horizontal line $y = 2$.



Modeling Horizontal Motion

The position of a particle on the horizontal line $y = 2$ is given by $x(t) = 4t^3 - 16t^2 + 15t$.

1. Graph the following parametric equations:
   a. $x_1(t) = 4t^3 - 16t^2 + 15t$
   b. $y_1(t) = 2$
   c. $[-4, 6]$ $[-3, 5]$

2. Graph the following parametric equations in the same window
   a. $x_2(t) = x_1(t)$
   b. $y_2(t) = t$
3. Graph the following parametric equations in the same window
   a. $x_3(t) = t$
   b. $y_3(t) = x_1(t)$

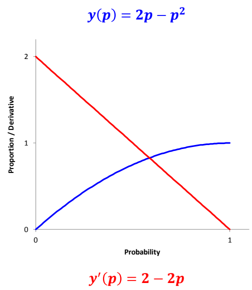
Seeing Motion on a Graphing Calculator

Simulate the motion of the rock from Example 4.

1. Graph the following parametric equations:
   a. $x_1(t) = 3(t < 5) + 3.1(t \geq 5)$
   b. $y_1(t) = 160t - 16t^2$
   c. What should we use for the limits on t?
   d. $[0, 6]$ $[-10, ?]$

Sensitivity to Change

- Gregor Mendel showed
  - $y = 2p - p^2$
- When is y is most sensitive to changes in p?



Exercise 5
Draining a Tank

A tank drains in 12 hours

$$Depth = y = 6\left(1 - \frac{t}{12}\right)^2$$

a) Find $\frac{dy}{dt}$

b) When is the level falling fastest? Slowest? What are the values of dy/dt?

c) Graph y and dy/dt together. Compare their behavior.

Economics

- Derivatives in economics are called ***marginals***.
- The cost of production is a function of the number of units produced, $c(x)$.
- The marginal cost of production is dc/dx.

$$\frac{dc}{dx} = \lim_{h\to 0} \frac{c(x+h) - c(x)}{h}$$

$$\frac{\Delta c}{\Delta x} = \frac{c(x+1) - c(x)}{1}$$

Example 6
Marginal Cost / Marginal Revenue

- Suppose the cost to produce *x* radiators is
  $$c(x) = x^3 - 6x^2 + 15x$$
- The revenue from selling *x* radiators is
  $$r(x) = x^3 - 3x^2 + 12x$$
- Find the marginal cost for selling 10 radiators.

$$c'(x) = \frac{d}{dx}(x^3 - 6x^2 + 15x) = 3x^2 - 12x + 15$$
$$c'(10) = 3(10)^2 - 12(10) + 15 = 195$$

Example 6
Marginal Cost / Marginal Revenue

- Find the marginal revenue.

$$r'(x) = \frac{d}{dx}(x^3 - 3x^2 + 12x)$$
$$= 3x^2 - 6x + 12$$

$$r'(10) = 3(10)^2 - 6(10) + 12 = 252$$

Exercise 6

Suppose the weekly revenue in dollars from selling custom-made desks is

$$r(x) = 2000\left(1 - \frac{1}{x+1}\right).$$

a) Draw a graph of r. What values of x make sense?

b) Find the marginal revenue when x desks are sold.

c) Use $r'(x)$ to estimate the increase revenue of increasing sales from 5 to 6 desks per week.

d) Find $\lim_{x\to\infty} r'(x)$. Interpret this number.

Homework

p 135: 1, 9, 10, 14, 15, 18, 19, 21, 24, 25, 27, 32, 34, 35, 38, 47, 49, 50