Chapter 3.3

Rules for Differentiation

Objectives

- Use the rules of differentiation.
- Find second and higher order derivatives.
- Use the derivative to calculate the instantaneous rate of change.

Learning Target

- 80% of the students will be able to find the derivative the function $f(x) = 2x^3 - 5x$.

Standard

G-C.4 Construct a tangent line from a point outside a given circle to the circle.

Overview

- Positive Integer Powers
- Multiples, Sums, and Differences
- Products and Quotients
- Negative Integer Powers of x
- Second and Higher Order Derivatives

Derivative of a Constant

$f(x) = c$ c is a constant.

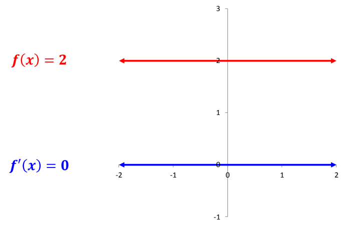
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{c - c}{h}$$

$$= 0$$

The derivative of a constant is 0.

Derivative of a Constant



Derivative of an Integer Power

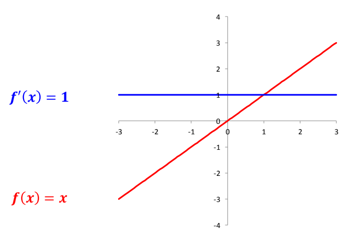
$f(x) =$x

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{x + h - x}{h}$$

$$= \lim_{h \to 0} \frac{h}{h} = 1$$

Derivative of an Integer Power

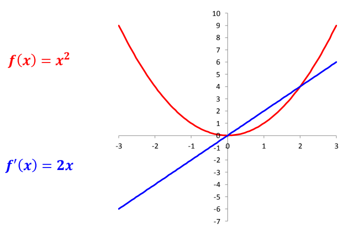


Derivative of an Integer Power

$f(x) = x^2$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$
$$= \lim_{h\to 0} \frac{x^2 + 2hx + h^2 - x^2}{h}$$
$$= \lim_{h\to 0} \frac{2hx + h^2}{h} = \lim_{h\to 0}(2x + h) = 2x$$

Derivative of an Integer Power

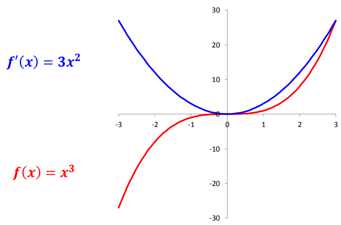


Derivative of an Integer Power

$f(x) = x^3$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h}$$
$$= \lim_{h\to 0} \frac{x^3 + 3hx^2 + 3h^2x + h^3 - x^3}{h}$$
$$= \lim_{h\to 0} \frac{3hx^2 + 3h^2x + h^3}{h}$$
$$= \lim_{h\to 0}(3x^2 + 3hx + h^2) = 3x^2$$

Derivative of an Integer Power

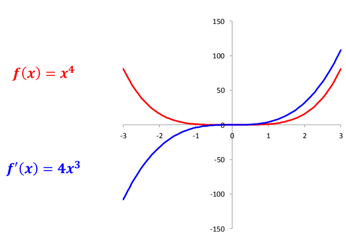


Derivative of an Integer Power

$$f(x) = x^4$$

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{(x+h)^4 - x^4}{h} \\ &= \lim_{h\to 0}\frac{x^4 + 4hx^3 + 6h^2x^2 + 3h^3x + h^4 - x^4}{h} \\ &= \lim_{h\to 0}\frac{4hx^3 + 6h^2x^2 + 4h^3x + h^4}{h} \\ &= \lim_{h\to 0}(4x^3 + 6hx^2 + 4h^2x + h^3) = 4x^3 \end{aligned}$$

Derivative of an Integer Power



Derivative of an Integer Power

$f(x) = x^n$ n is a positive integer.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$
$$= \lim_{h \to 0} \frac{x^n + nhx^{n-1} + \cdots - x^n}{h}$$
$$= \lim_{h \to 0} \frac{nhx^{n-1} + \cdots}{h}$$
$$= \lim_{h \to 0} (nx^{n-1} + \cdots) = nx^{n-1}$$

Derivative of an Integer Power

The derivative of a power function is

$$\frac{d}{dx}(x^n) = nx^{n-1},$$

where n is a positive integer.

Derivative of a Constant Multiplier

$f(x)$ is a differentiable function at x.

$$\frac{d}{dx}(cf(x)) = \lim_{h \to 0} \frac{cf(x+h) - cf(x)}{h}$$
$$= c \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= c\, f'(x)$$

The derivative of a function multiplied by a constant is the constant times the derivative of the function.

Derivative of a Sum or Difference

$f(x)$ and $g(x)$ are a differentiable functions at x.

$$\frac{d}{dx}\big(f(x) \pm g(x)\big) = \lim_{h\to 0}\frac{f(x+h) \pm g(x+h) - \big(f(x) \pm g(x)\big)}{h}$$

$$= \lim_{h\to 0}\left[\frac{\big(f(x+h) - f(x)\big)}{h}\right] \pm \lim_{h\to 0}\left[\frac{\big(g(x+h) - g(x)\big)}{h}\right]$$

$$= f'(x) \pm g'(x)$$

The derivative of sum or difference of two functions is the sum or difference of the derivatives of the functions.

Example 1

Find $\frac{dy}{dx}$ if $y = x^3 + 6x^2 - \frac{5}{3}x + 16$

$$\frac{dy}{dx} = \frac{d}{dx}(x^3) + \frac{d}{dx}(6x^2) - \frac{d}{dx}\left(\frac{5}{3}x\right) + \frac{d}{dx}(16)$$

$$= 3x^2 + 6 \cdot 2x - \frac{5}{3} + 0$$

$$= 3x^2 + 12x - \frac{5}{3}$$

Exercise 1

Find $\frac{dy}{dx}$ if

$$y = \frac{x^3}{3} + \frac{x^2}{2} + x$$

Example 2

Find the horizontal tangents of $y = x^4 - 2x^2 + 2$

$$\frac{dy}{dx} = 0$$

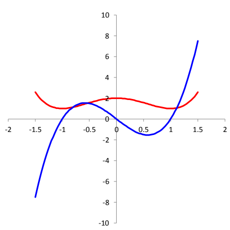
$$\frac{dy}{dx} = 4x^3 - 4x = 0$$

$$4x(x^2 - 1) = 0$$

$$x = 0, -1, 1$$

Example 2

$f(x) = x^4 - 2x^2 + 2$



$f'(x) = 4x^3 - 4x$

Exercise 2

Find the horizontal tangents of

$$y = x^3 - 2x^2 + x + 1$$

Example 3

Find the horizontal tangents of

$$y = 0.2x^4 - 0.7x^3 - 2x^2 + 5x + 4$$

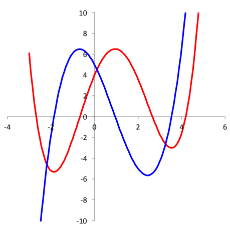
$$\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = 0.8x^3 - 2.1x^2 - 4x + 5 = 0$$

Example 3

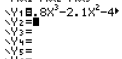
$\boldsymbol{f(x) = 0.2x^4 - 0.7x^3 - 2x^2 + 5x + 4}$

$\boldsymbol{f'(x) = 0.8x^3 - 2.1x^2 - 4x + 5}$



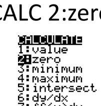
Using a Graphing Calculator

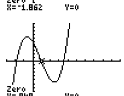
Enter the derivative of the function into the calculator.

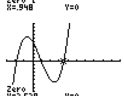


Use the solver to find zeros:

2^{nd} CALC 2:zero

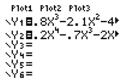




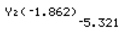


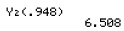
Using a Graphing Calculator

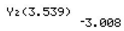
Enter the function into the calculator.



Use the calculator to calculate $f(x)$.







Exercise 3

Find the horizontal tangents of

$$y = x^4 - 7x^3 + 2x^2 + 15$$

Derivative of a Product

Does

$$\frac{d}{dx}\big(f(x) \cdot g(x)\big) = \frac{d}{dx}\big(f(x)\big) \cdot \frac{d}{dx}\big(g(x)\big)$$

$$\frac{d}{dx}(x^2) = 2x \qquad \frac{d}{dx}(x) \cdot \frac{d}{dx}(x) = 1 \cdot 1$$

NO!

Derivative of a Product

$f(x)$ and $g(x)$ are a differentiable functions at x.

$$\frac{d}{dx}\big(f(x)g(x)\big) = \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)g(x+h) + f(x+h)g(x) - f(x+h)g(x) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)\big(g(x+h) - g(x)\big) + g(x)\big(f(x+h) - f(x)\big)}{h}$$

$$= \lim_{h\to 0} f(x+h)\cdot\frac{g(x+h) - g(x)}{h} + \lim_{h\to 0} g(x)\cdot\frac{f(x+h) - f(x)}{h}$$

Derivative of a Product

$$= \lim_{h\to 0} f(x+h)\cdot\frac{g(x+h) - g(x)}{h} + \lim_{h\to 0} g(x)\cdot\frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} f(x+h)\cdot\lim_{h\to 0}\frac{g(x+h) - g(x)}{h}$$
$$+ g(x)\cdot\lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

$$= f(x)\cdot g'^{(x)} + g(x)\cdot f'(x)$$

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx}$$

Derivative of a Product

The derivative of product of two functions is the product of the first function and the derivative of the second function added to product of the second function and the derivative of the first function.

Example 4

Find $f'(x)$ if $f(x) = (x^2 + 1)(x^3 + 3)$

$$f'(x) = \frac{d}{dx}\big((x^2 + 1)(x^3 + 3)\big)$$

$$= (x^2 + 1)(3x^2) + (x^3 + 3)(2x)$$

$$= 3x^4 + 3x^2 + 2x^4 + 6x$$

$$= 5x^4 + 3x^2 + 6x$$

Exercise 4

Let $y = (x - 4)(x^3 - 1)$

a. Find $f'(x)$ using the product rule

b. Find $f'(x)$ by multiplying the factors and then differentiating.

c. What if $f(x) = x^2 \sin x$?

Derivative of a Quotient

$f(x)$ and $g(x)$ are a differentiable functions at x.

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \lim_{h \to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}$$

$$= \lim_{h \to 0} \frac{g(x)f(x+h) - f(x)g(x+h)}{hg(x+h)g(x)}$$

$$= \lim_{h \to 0} \frac{g(x)f(x+h) - g(x)f(x) - f(x)g(x+h) + g(x)f(x)}{hg(x+h)g(x)}$$

Derivative of a Quotient

$$\left(\frac{f}{g}\right)' = \lim_{h\to 0}\frac{g(x)f(x+h) - g(x)f(x) - f(x)g(x+h) + g(x)f(x)}{hg(x+h)g(x)}$$

$$= \lim_{h\to 0}\frac{g(x)\frac{f(x+h)-f(x)}{h} - f(x)\frac{g(x+h)-g(x)}{h}}{g(x+h)g(x)}$$

$$= \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

$$\boldsymbol{\left(\frac{f}{g}\right)' = \frac{g\cdot f' - f\cdot g'}{g^2}}$$

Example 5

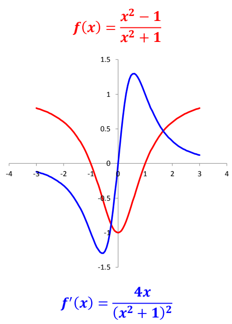
Differentiate

$$f(x) = \frac{x^2-1}{x^2+1}$$

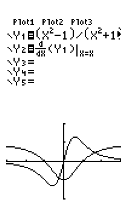
$$f' = \frac{(x^2+1)2x - (x^2-1)2x}{(x^2+1)^2}$$

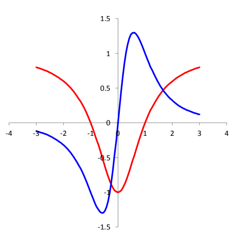
$$= \frac{2x^3+2x-2x^3+2x}{(x^2+1)^2}$$

$$= \frac{4x}{(x^2+1)^2}$$



Use a Graphing Calculator





Exercise 5

Find the derivative of

$$f(x) = (1 - x)(1 + x^2)^{-1}$$

Example 6

Let $y = f(x)g(x)$. Find $y'(2)$ if

$f(2) = 3$, $f'(2) = -4$, $g(2) = 1$, and $g'(2) = 2$

Use the Product Rule: $y' = (fg)' = fg' + f'g$

$$\begin{aligned} y'^{(2)} &= f(2)g'(2) + f'(2)g(2) \\ &= (3)(2) + (-4)(1) \\ &= 2 \end{aligned}$$

Exercise 6

u and v are functions that are differentiable at $x = 0$, and $u(0) = 5$, $u'(0) = -3$, $v(0) = -1$, and $v'(0) = 2$. Find the following derivatives:

a. $\frac{d}{dx}(uv)$

b. $\frac{d}{dx}\left(\frac{u}{v}\right)$

a. $\frac{d}{dx}\left(\frac{v}{u}\right)$

b. $\frac{d}{dx}(7v - 2u)$

Negative Integer Powers

Suppose n is a negative integer.
Let $n = -m$.

$$x^n = x^{-m} = \frac{1}{x^m}$$

$$\frac{d}{dx}(x^n) = \frac{d}{dx}\left(\frac{1}{x^m}\right) = \frac{x^m \frac{d}{dx}(1) - 1 \cdot \frac{d}{dx}(x^m)}{(x^m)^2}$$

$$= \frac{0 - mx^{m-1}}{x^{2m}} = -m\frac{1}{x^{m+1}} = -mx^{-m-1} = nx^{n-1}$$

Negative Integer Powers

If n is a negative integer, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

The power rule applies to all integer powers.

Example 7

$$f(x) = \frac{x^2+3}{2x}$$

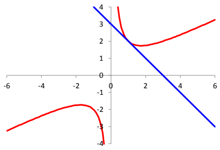
Find $f'(x)$ at $(1, 2)$.

We could use the Quotient Rule, but the Power Rule is easier.

$$f(x) = \frac{1}{2}x + \frac{3}{2}x^{-1}$$

Example 7

$$f'(x) = \frac{d}{dx}\left(\frac{1}{2}x + \frac{3}{2}x^{-1}\right)$$
$$= \frac{1}{2} - \frac{3}{2}x^{-2}$$
$$f'(1) = \frac{1}{2} - \frac{3}{2} = -1$$
$$y - 2 = -1(x - 1)$$
$$y = -x + 3$$



Exercise 7

Find a line tangent to
$$y = \frac{x^3 + 1}{2x}$$
at $x = 1$

Higher Order Derivatives

- First derivative of y
  - $y' = \frac{dy}{dx}$
- Second derivative of y
  - $y'' = \frac{dy'}{dx} = \frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{d^2y}{dx^2}$
- Third derivative of y
  - $y''' = \frac{dy''}{dx} = \frac{d\left(\frac{d^2}{dx^2}\right)}{dx} = \frac{d^3y}{dx^3}$

Higher Order Derivatives

- n^{th} Derivative
  - $y^{(n)} = \frac{d}{dx} y^{(n-1)} = \frac{d^n y}{dx^n}$

Do not confuse the n^{th} derivative of y ($y^{(n)}$) with the n^{th} power of y (y^n).

Example 8

Find the first four derivatives of $y = x^3 - 5x + 2$

1. $y' = 3x^2 - 5$
2. $y'' = 6x$
3. $y''' = 6$
4. $y^{(4)} = 0$

Exercise 8

Find the first four derivatives of
$$y = x^4 + x^3 - 2x^2 + x - 5$$

Example 9

An orange farmer currently has 200 trees yielding an average of 15 bushels of oranges per tree. She is expanding her farm at the rate of 15 trees per year. Improved husbandry is improving her average yield by 1.2 bushels per tree. What is the current (instantaneous) rate of increase of her total annual production of oranges?

Example 9

Define the following functions:
$T(x) = number\ of\ trees\ x\ years\ from\ now$
$Y(x) = yield\ per\ tree\ x\ years\ from\ now$
Therefore, the total production in *x* years is,
$P(x) = T(x)Y(x)$
We know:

$T(0) = 200 \qquad Y(0) = 15$
$T'(0) = 15 \qquad Y'(0) = 1.2$

Example 9

Using the Product Rule,
$$P'(x) = T(x)Y'(x) + Y(x)T'(x)$$
At $x = 0$,
$$P'(0) = T(0)Y'(0) + Y(0)T'(0)$$
Therefore,
$$P'(0) = (200)(1.2) + (15)(15)$$
$P'(0) = 465$ bushels per year

Exercise 9

An apple farmer currently has 156 trees yielding an average of 12 bushels of apples per tree. He is expanding his farm at the rate of 13 trees per year. Improved husbandry is improving his average yield by 1.5 bushels per tree. What is the current (instantaneous) rate of increase of his total annual production of apples? Answer in appropriate units of measure.

Homework

P 124: 3-45, multiples of 3