Chapter 3.2

Differentiability

Objectives

- Find where a function is not differentiable.
- Distinguish among corners, cusps, discontinuities, and vertical tangents.
- Approximate derivatives numerically.
- Approximate derivatives graphically.

Learning Target

- 80% of the students will be able to determine whether or not the function $f(x) = \sqrt[3]{x}$ is differentiable at $x = 0$.

Standard

G-C.4 Construct a tangent line from a point outside a given circle to the circle.

Overview

- How a Derivative Might Not Exist
- Local Linearity
- Derivatives on a Calculator
- Differentiability and Continuity
- Intermediate Value Theorem for Derivatives

How Might a Derivative Fail to Exist?

A Function will not have a derivative at a point $P(a, f(a))$ where the slopes of the secant lines

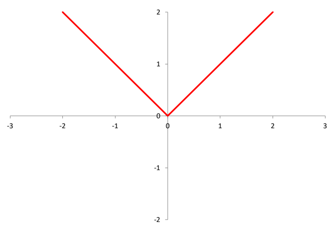
$$\frac{f(x) - f(a)}{x - a}$$

Fail to approach a limit as x approaches a.

A function whose graph is otherwise smooth will fail to have a derivative where the graph has a ***corner***, *cusp*, *vertical tangent*, or *discontinuity*.

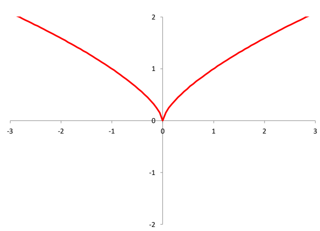
Corner

$f(x) = |x|$



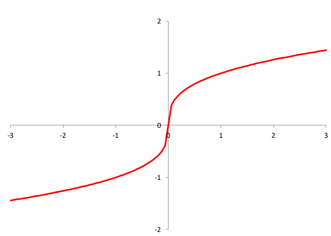
Cusp

$f(x) = x^{2/3}$



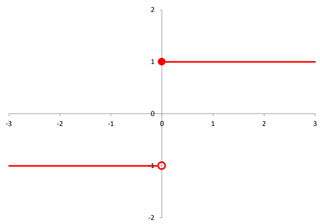
Vertical Tangent

$f(x) = \sqrt[3]{x}$



Discontinuity

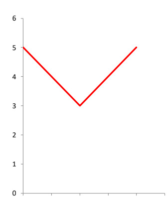
$$f(x) = \begin{cases} -1 \Leftarrow x < 1 \\ 1 \Leftarrow x \geq 1 \end{cases}$$



Example 1

Find all points in the domain of
$f(x) = |x - 2| + 3$
where f is not differentiable.

- Think Graphically.
- $f(x) = 2$ units right and three units up.
- Corner at $(2, 3)$.
- Not differentiable.



Exercise 1

Compare the left-hand and right-hand derivatives to show that

$$f(x) = \begin{cases} 2 \Leftarrow x \leq 1 \\ 2x \Leftarrow > 1 \end{cases}$$

to show that the function is not differentiable at $(1, 2)$.

Local Linearity

A function that is differentiable at a point a more closely resembles its tangent line as we "zoom in" on $(a, f(a))$.

Consider $f(x) = x\cos(3x)$.

[–4, 4] by [–3, 3]   [1.7, 2.3] by [1.7, 2.1]   [1.93, 2.07] by [1.85, 1.95]

Exploration 1

Consider

$f(x) = |x| + 1$

$g(x) = \sqrt{x^2 + 0.0001} + 0.99$

1. Graph f and zoom in on the point (0, 1). Does the corner show signs of straightening out?
2. Graph g and zoom in on the point (0, 1). Does the corner show signs of straightening out?

3. How many zooms does it take before the graph of g looks exactly like a horizontal line?
4. Graph f and g together in a square window. Do they look the same? Zoom in. Does f change? What about g?

Symmetric Difference Quotient

For small values of h, the difference quotient

$$\frac{f(a+h)-f(a)}{h}$$

often gives a good approximation of $f'(a)$.

The **symmetric difference quotient**

$$\frac{f(a+h)-f(a-h)}{2h}$$

usually gives a better approximation of $f'(a)$.

Example 2
Finding Derivatives on a Calculator

Graphing calculators usually use the symmetric difference quotient to approximate the derivative of a function.

Let's find $f'(2)$ if $f(x) = x^2$.

$\frac{d}{dX}(X^2)|_{X=2}$
4

Now find the symmetric difference quotient with $h = 0.001$

$(2.001^2-1.999^2)$
.008
Ans/0.002
4

Exercise 2

Is $f(x) = 4x - x^2$ differentiable at $x = 3$?

Use the symmetric difference quotient to find $f'(3)$. Use $h = 0.001$.

Example 3

Use a graphing calculator to compute the numerical of $f(x) = |x|$ at $x = 0$.

$\frac{d}{dX}(|X|)|_{X=0}$ $\quad 0$

This is incorrect, because $f(x) = |x|$ is not differentiable at $x = 0$.

Exercise 3

Is $f(x) = x^{4/5}$ differentiable at $x = 0$?
Use the symmetric difference quotient to find $f'(0)$. Use $h = 0.001$.

Exploration 2

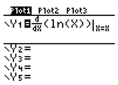
Let $f(x) = x^2$ and $h = 0.01$.

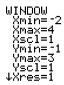
1. Find
   $$\frac{f(10+h) - f(10)}{h}$$
   How close is it to $f'(10)$?
2. Find
   $$\frac{f(10+h) - f(10-h)}{2h}$$
   How close is it to $f'(10)$?
3. Repeat for $f(x) = x^3$

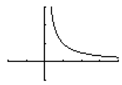
Example 4

Graphing a Derivative

Let $f(x) = \ln x$







Can you tell by analyzing the graph what $f'(x)$ is?

Exercise 4

Use your graphing calculator to graph the derivative of $y = -\cos x$.

What is this function?

Differentiability and Continuity

- If a function is differentiable at $x = c$, then the function is continuous at $x = c$.
- The contrapositive is also true:
  - If a function is not continuous at $x = c$, then the function is not differentiable at $x = c$.
- The converse is not always true:
  - If a function is continuous at $x = c$, then the function is differentiable at $x = c$.
  - Counter-example - $f(x) = |x - 1|$ at $x = 1$.

Intermediate Value Theorem for Derivatives

If a and b are any two points in an interval on which f is differentiable, then f' takes on every value between $f'(a)$ and $f'(b)$.

Example 5

Does any function have the unit step function as its derivative?

Choose some $a < 0$ and some $b > 0$.

$$U(a) = 0$$
$$U(b) = 1$$

U does not take on any values between 0 and 1.

By the Intermediate Value Theorem for Derivatives, U cannot be a derivative.

Exercise 5

Show that the function

$$f(x) = \begin{cases} x & \Leftarrow x < 1 \\ 2x & \Leftarrow x \leq 1 \end{cases}$$

is not the derivative of any function on the interval $0 \leq x \leq 2$.

Homework

P 114: 1-25, odd