Chapter 3.1

Derivatives

Objectives

- Definition of a Derivative
- Notation
- Relationships between the Graphs of f and f'
- Graphing the Derivative from Data
- One-sided Derivatives

Learning Target

- 80% of the students will be able to find the derivative $\frac{d}{dx}(x^2)$.

Standard

G-C.4 Construct a tangent line from a point outside a given circle to the circle.

Overview

- Calculate slopes and derivatives using the definition of the derivative.
- Graph f from the graph of f'.
- Graph f' from the graph of f.
- Graph the derivative of a function given numerically with data.

The Derivative and the Tangent Line Problem

- The slope of the tangent line of $f(x)$ at the point $(x, f(x))$ is $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.
- Definition:

The derivative of f at x is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided that the limit exists.

Example 1

Differentiate $f(x) = x^3$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h}$$
$$= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$$
$$= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$$

Example 1

$$= \lim_{h\to 0} \frac{(3x^2 + 3xh + h^2)h}{h}$$

$$= \lim_{h\to 0}(3x^2 + 3xh + h^2)$$

$$= 3x^2$$

Exercise 1

Find the derivative of $f(x) = x^2 + 4$ at $x = 1$.

Alternate form of the Derivative

- The derivative of f at $x = a$ is given by:
- $\boldsymbol{f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}}$
- To show this limit exists, show that the derivative from the left equals the derivative from the right:
- $\boldsymbol{\lim_{x \to a^-} \frac{f(x)-f(a)}{x-a} = \lim_{x \to a^+} \frac{f(x)-f(a)}{x-a}}$

Example 2

Differentiate $f(x) = \sqrt{x}$ using the alternate definition.

At the point $x = a$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to a} \frac{\sqrt{x} - \sqrt{a}}{x - a}$$

$$= \lim_{x \to a} \frac{\sqrt{x} - \sqrt{a}}{(\sqrt{x} - \sqrt{a})(\sqrt{x} + \sqrt{a})}$$

Example 2

$$= \lim_{x \to a} \frac{1}{\sqrt{x} + \sqrt{a}}$$

$$= \frac{1}{2\sqrt{a}}$$

$$f'(x) = \frac{1}{2\sqrt{x}} \qquad \forall x > 0$$

Exercise 2

Use the alternative form of the derivative to differentiate $f(x) = x^2 + 4$ at $x = 1$.

Notations for the Derivative

$f'(x)$ pronounced "f prime of x" (use when the function is written $f(x) =$).

$\frac{dy}{dx}$ the derivative of y with respect to x (use when the function is written $y =$).

$\frac{df}{dx}$ emphasizes the name of the function.

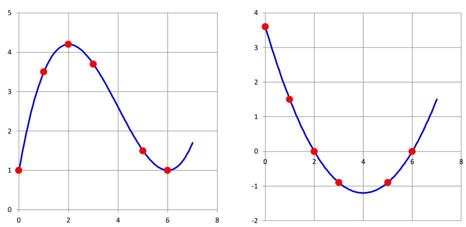
$\frac{d}{dx}(f(x))$ (use when the function is written $f(x) =$).

Derivative

- The derivative of $y = f(x)$ at x is the slope of the tangent line at the point $(x, f(x))$.
- This is also the slope of the function $y = f(x)$ at x.
- The slope refers to the rate of change of y with respect to a change in x.
- The point-slope form of the equation of a line with slope m passing through the point (x_1, y_1) is $y - y_1 = m(x - x_1)$.

Example 3

Estimate the derivative from slope of a graph.



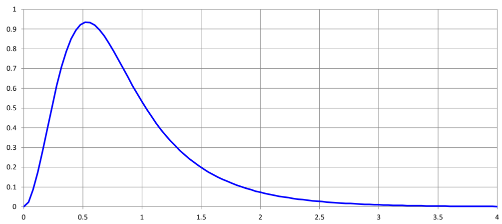
Suppose the first graph represents the depth of water in a ditch over a 7-day period.

- What does the second graph represent?
- Describe what happened to the water over this period.
- Describe the weather during this period.
- How does the graph of the derivative help in finding when the weather was wettest or driest?

- Interpret the significance of the point on day 2 in terms of the water in the ditch.
- How does the corresponding point in the second graph reflect that in terms of rate of change?
- It is tempting to conclude that the rain stops at the beginning of day 2. Explain why that may not be true,

Exercise 3

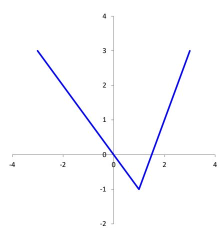
Estimate the slope on different points on this graph and draw a graph of the derivative.



Example 4

Sketch a graph of the function that has the following properties:

- $f(0) = 0$
- $f'(x) = \begin{cases} -1 & \Leftarrow x < 1 \\ 2 & \Leftarrow x > 1 \end{cases}$
- f is continuous for all x



Exercise 4

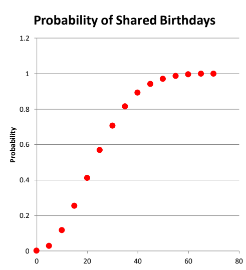
Sketch a graph of the function that has the following properties:

- $f(0) = 1$
- $f'(x) = \begin{cases} 2 & \Leftarrow x < 2 \\ -1 & \Leftarrow x > 2 \end{cases}$
- f is continuous for all x

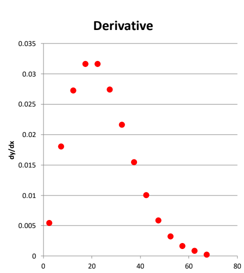
Example 5

What is the probability that two people in this class have the same birthday?

Probability of shared Birthdays	
Number of People	Probability
0	0
5	0.027
10	0.117
15	0.253
20	0.411
25	0.569
30	0.706
35	0.814
40	0.891
45	0.941
50	0.970
55	0.986
60	0.994
65	0.998
70	0.999



Estimates of Slope	
Midpoint	Dy/Dx
2.5	0.0054
7.5	0.018
12.5	0.0272
17.5	0.0316
22.5	0.0316
27.5	0.0274
32.5	0.0216
37.5	0.0154
42.5	0.01
47.5	0.0058
52.5	0.0032
57.5	0.0016
62.5	0.0008
67.5	0.0002



Exercise 5

Use the table to the right to sketch the derivative and answer the following:

a. What does the derivative represent?
b. What are the units of the derivative?
c. Can you guess an equation for the derivative from its graph?

Skiing Distances	
T (sec)	D (feet)
0	0
1	3.3
2	13.3
3	29.9
4	53.2
5	83.2
6	119.8
7	163.0
8	212.9
9	269.5
10	332.7

One-Sided Derivatives

A function $y = f(x)$ is **differentiable on a closed interval** $[\boldsymbol{a}, \boldsymbol{b}]$ if it has a derivative at every interior point of the interval, and if the limits

$$\lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$$
$$\lim_{h \to 0^-} \frac{f(b+h) - f(b)}{h}$$

exist at the endpoints.

- In the right-hand derivative, h is positive and $a + h$ approaches a from the right.
- In the left-hand derivative, h is negative and $b + h$ approaches b from the left.
- Right-hand and left-hand derivatives may be defined at any point of a function's domain.
- A function has a two-sided derivative if and only its left and right-aanded derivatives are equal.

Example 6

Show that the following function has left-handed and right-handed derivatives at $x = 0$, no derivatives there.

$$y = \begin{cases} x^2, if\ x \leq 0 \\ 0, if\ x > 0 \end{cases}$$

$$\lim_{h\to 0^-} \frac{(0+h)^2 - 0^2}{h} = \lim_{h\to 0^-} \frac{h^2}{h} = 0$$

$$\lim_{h\to 0^+} \frac{2(0+h) - 0^2}{h} = \lim_{h\to 0^-} \frac{2h}{h} = 2$$

Since the left hand and right hand derivatives are unequal, the function has no derivative at $x = 0$.

Homework

P 105: 1-35, odd