Chapter 2.4

Rates of Change and Tangent Lines

Objectives

- Average Rates of Change
- Tangent to a Curve
- Slope of a Curve
- Normal to a Curve
- Speed Revisited

Learning Target

- 80% of the students will be able to write the equation for the line tangent to the parabola $y = x^2$ at the point $P(2, 4)$.

Standard

G-C.4 Construct a tangent line from a point outside a given circle to the circle.

Overview

- Apply directly the definition of the slope of a curve in order to calculate slopes..
- Find the equations of the tangent line and normal line to a curve at a given point.
- Find the average rate of change of a function.

Rate of Change

- Mile post 282
- Mile post 242
- Elapsed time – 32 min
- Average speed?
  - $\frac{282\ mi - 242\ mi}{32\ min} \times \frac{60\ min}{h}$
  - 75 mph
- Instantaneous speed?

Average Rate of Change

- Find the average rate of change of the function $f(x) = x^3 - x$ over [1, 3].
  - $f(1) = 0$
  - $f(3) = 24$
  - $\frac{f(3)-f(1)}{3-1} = \frac{24-0}{2} = 12$

Exercise 1

Find the average rate of change of

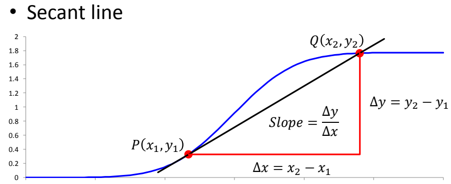
$$f(x) = \sqrt{4x + 1}$$

a. [0, 2]

b. [10, 12]

Average Slope

- Slope of a line drawn through two points on a curve.
  - Secant line



Example 2

- Find the average rate of change of $f(x) = \ln x$ over the given intervals.

a. [1, 4]

$$\frac{\ln 4 - \ln 1}{4 - 1} = \frac{\ln \frac{4}{1}}{3} = \frac{\ln 4}{3} \approx 0.462$$

b. [100, 103]

$$\frac{\ln 103 - \ln 100}{103 - 100} = \frac{\ln \frac{103}{100}}{3} = \frac{\ln 1.03}{3} \approx 0.0099$$

Exercise 2

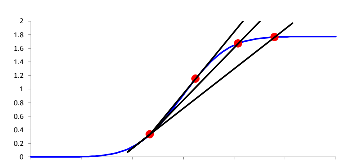
Find the average rate of change of the function $f(x) = 2 + \cos x$ over the given intervals.

1. $[0, \pi]$

2. $[-\pi, \pi]$

Rate of Change

- The average rate of change is the secant line between two points on the curve.
- What if the two points come closer and closer together?



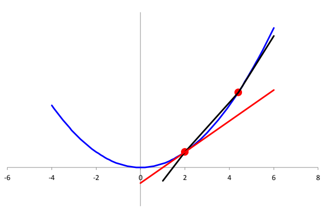
Tangent Line

As the two points come closer and closer together, the secant line comes closer and closer to a line that is tangent to the curve.

1. Calculate the slope of the secant through the two points.
2. Find the limiting value as the two points approach each other.
3. The slope of the curve is the slope of the tangent.

Example 3

Find the slope of the parabola $y = x^2$ at the point $P(2, 4)$.



Example 3

1. Find the slope of the secant line.
$$\frac{\Delta y}{\Delta x} = \frac{(2+h)^2 - 4}{h} = \frac{4 + 4h + h^2 - 4}{h} = 4 + h$$
2. Find the limit as $h \to 0$.
$$\lim_{h\to 0} 4 + h = 4$$
3. Find the equation of the line passing through $P(2, 4)$ that has this slope.
$$y - 4 = 4(x - 2)$$
$$y = 4x - 4$$

Exercise 3

Find the slope of $y = x^2 - 4x$ and an equation of the tangent line at $x = 1$.

Slope of a Curve

To find the slope of a curve $y = f(x)$ at the point $P(a, f(a))$:

1. Calculate the slope of the secant line through P and a point $Q(a + h, f(a + h))$.
2. Try to find the limit as $h \to 0$.
3. If the limit exists, it is the slope of the curve at P.
4. Define the tangent at P as the line through P with this slope.

The Slope of a Curve at P

The **slope of the curve** $y = f(x)$ at the point $P(a, f(a))$ is the number

$$m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h},$$

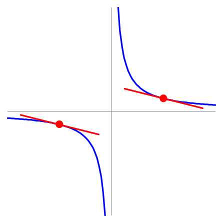
Provided the limit exists.

The **tangent line to the curve** at P is the curve through P with this slope.

Example 4

$f(x) = \frac{1}{x}$

a. Find the slope of the curve at $x = a$.
b. Where does the slope equal $-{}^1/_4$?
c. What happens to the tangent to the curve at the point $(a, {}^1/_a)$ for different values of a?



Example 4 – Solution

a. The slope at $x = a$ is

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h}$$

$$= \lim_{h\to 0} \frac{1}{h} \cdot \left[\frac{1}{a+h} - \frac{1}{a}\right]$$

$$= \lim_{h\to 0} \left[\frac{a - (a+h)}{ha(a+h)}\right] = \lim_{h\to 0} \frac{-h}{ha(a+h)}$$

$$= \lim_{h\to 0} \frac{-1}{a(a+h)} = -\frac{1}{a^2}$$

Example 4 – Solution

b. The slope will be $-{}^1/_4$ if

$$-\frac{1}{a^2} = -\frac{1}{4}$$

$$a^2 = 4$$

$$a = \pm 2$$

The curve has the slope $-{}^1/_4$ at two points $(2, {}^1/_2)$ and $(-2, -{}^1/_2)$.

Example 4 – Solution

c. The slope, $m = -\frac{1}{a^2}$, is always negative.

1. As $a \to 0^+$ the slope approaches $-\infty$. The tangent becomes increasingly steep.
2. Moreover, as $a \to 0^-$ the slope approaches $-\infty$.
3. As $a \to \pm\infty$ the slope approaches 0. The tangent becomes increasingly horizontal.

Exercise 4

$y = 9 - x^2$

a. Find the slope of the curve at $x = a$

b. Describe what happens to the tangent at $x = a$ as a changes.

Difference Quotient

$$\frac{f(a+h) - f(a)}{h}$$

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Foundation of differential calculus

Normal to a Curve

The **normal line** to a curve at a point P is the line that is perpendicular to the tangent at that point.

1. Find the slope of the tangent line at P.
2. $m_{\perp} = -\frac{1}{m_{\parallel}}$.
3. $y - f(a) = m_{\perp}(x - a)$

Example 5

Write an equation for the normal to the curve $f(x) = 4 - x^2$ at $x = 1$.

1. Slope

$$\lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{4-(1+h)^2-3}{h}$$

$$= \lim_{h\to 0}\frac{4-1-2h-h^2-3}{h}$$

$$= \lim_{h\to 0}\frac{-2h-h^2}{h} = -2$$

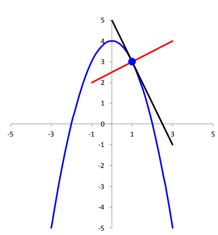
Example 5

2. The slope of the normal is,

$$m_{\perp} = -\frac{1}{m_{\parallel}} = \frac{1}{2}$$

The normal to the curve at $(1, f(1)) = (1, 3)$ is,

$$y - 3 = \frac{1}{2}(x - 1)$$



Exercise 5

For $y = x^2 - 4x$ find an equation of the normal line at $x = 1$. (The slope at $x = 1$ is $m = -2$.) Then graph the curve, its tangent line, and its normal line in a square window.

Instantaneous Speed

- Position
  - $y = f(t)$
- Average speed
  - $\frac{\Delta y}{\Delta t} = \frac{f(t)-f(0)}{t-0}$
- Instantaneous speed
  - $\lim_{h \to 0} \frac{f(t+h)-f(t)}{h}$

Example 6

- Find the speed of a falling rock dropped from rest at $t = 1$. (Assume the rock was not moving at $t = 0$.)
  - $y = 16t^2$
  - $\frac{f(1+h)-f(1)}{h} = \frac{16(1+h)^2-16(1)^2}{h} = \frac{16+32h+16h^2-16}{h}$
    $= \frac{32h + h^2}{h} = 32 + 16h$
  - $\lim_{h \to 0}(32 + h) = 32\ ft/s$

Exercise 6

The equation for free fall on the surface of Jupiter is $y = 11.44t^2\ m$ with t in seconds. Assume a rock is dropped from a height of 500 m above the surface. Find the speed of the rock at $t = 2\ s$.

Homework

P 92: 1-33, odd