Chapter 2.3

Continuity

Objectives

- Continuity at a Point
- Continuous Functions
- Algebraic Combinations
- Composites
- Intermediate Value Theorem for Continuous Functions

Learning Target

80% of the students will be able to remove the discontinuity in the following function:

$$f(x) = \frac{x^3 - 3x^2 + x - 3}{x^2 + x - 12}$$

Standard

S-ID.6a Fit a function to the data; use functions fitted to data to solve problems in the context of the data. *Use given functions or choose a function suggested by the context. Emphasize linear, quadratic, and exponential models.*

Overview

- Understand the meaning of a continuous function.
- Identify the intervals over which a given function is continuous.
- Remove removable discontinuities by modifying a function.
- Apply the Intermediate Value Theorem.

Continuous

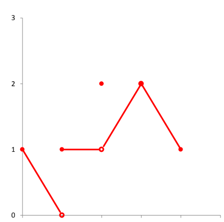
- Physical phenomena
  - Particle motion
  - Reaction rates
- Fit a curve to data
- Never “connect the dots”

Discontinuous

- Atomic transitions
  - Lasers
- Black body radiation
  - Ultraviolet “catastrophe”
  - Planck’s Law
  - Einstein – photons

Example 1
Continuity

- Consider this function over the interval $[0,4]$
- $\lim_{x\to 0^+} f(x) = f(0)$
- $\lim_{x\to 4^-} f(x) = f(4)$
- $\lim_{x\to 1} f(x)$ undefined
- $\lim_{x\to 2} f(x) = 1$



Continuity

- A function f is continuous at $x = c$ if **all** of the following conditions are true:

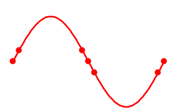
1. $f(c)$ is defined.
2. $\lim_{x\to c} f(x)$ exists.
3. $\lim_{x\to c} f(x) = f(c)$

Continuity at a Point

- Interior point:
  A function $y = f(x)$ is continuous at an interior point c of its domain if
  $$\lim_{x \to c} f(x) = f(c)$$
- End Point:
  A function $y = f(x)$ is continuous at an end point c of its domain if
  $$\lim_{x \to c^+} f(x) = f(c) \text{ or } \lim_{x \to c^-} f(x) = f(c)$$

Example 1, continued
Continuity at a point

- Continuity from the right
- Continuity from the left
- Two sided continuity



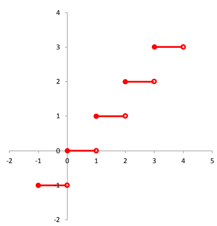
Exercise 1

Find the points of continuity and discontinuity on

$$y = \frac{1}{x^2 + 1}$$

Identify each type.

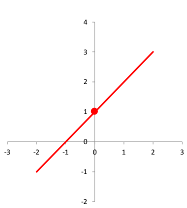
Example 2
Greatest Integer Function



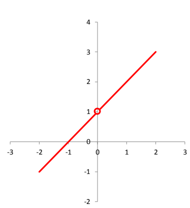
Discontinuities

- f has a removable discontinuity at $x = c$ when f is not continuous at $x = c$ AND $\lim_{x \to c} f(x)$ exists.
- f has a non-removable discontinuity at $x = c$ when $\lim_{x \to c} f(x)$ does not exist.

Continuous vs. Discontinuous

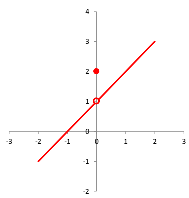


continuous

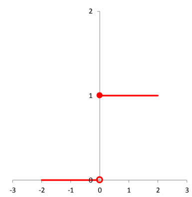


removable

Discontinuous



removable



jump

Exercise 2

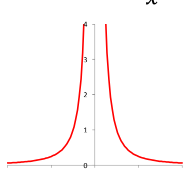
Find the points of continuity and discontinuity on

$$y = \frac{|x|}{x}$$

Identify each type.

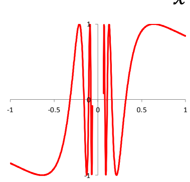
Discontinuous Functions

$$y = f(x) = \frac{1}{x^2}$$



$$\lim_{x \to 0} f(x) = \infty$$

$$y = f(x) = \sin \frac{1}{x}$$



$\lim_{x \to 0} f(x)$ undefined

Possible Discontinuities

- Check the following x-values:

1. Undefined values for a rational function
2. Vertical asymptotes for a rational or trigonometric function
3. Endpoints of intervals for a piecewise-defined function (**check one-sided limits**)

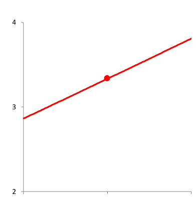
Removing a Discontinuity

$$f(x) = \frac{x^3 - 7x - 6}{x^2 - 9}$$

1. Factor the denominator. What is the domain of f?
   a. $(x+3)(x-3)$
   b. $\{x|x \neq \pm 3\}$

Removing a Discontinuity

2. Investigate the graph of f around $x = 3$.
3. How should f be defined at $x = 3$ to remove the discontinuity?
   a. $f(3) = \frac{10}{3}$



Removing a Discontinuity

4. Show that $(x - 3)$ is a factor of the numerator.
   a. $x^3 - 7x - 6 = (x - 3)(x^2 + 3x + 2)$
5. Remove all common factors and compute the limit as $x \to 3$

$$\lim_{x \to 3} \frac{x^2 + 3x + 2}{x + 3} = \frac{10}{3}$$

Removing a Discontinuity

6. The **extended** function

$$g(x) = \begin{cases} \dfrac{x^3 - 7x - 6}{x^2 - 9}, & x \neq 3 \\ \dfrac{10}{3}, & x = 3 \end{cases}$$

   is continuous at $x = 3$.
7. The function $g(x)$ is the **continuous extension** of $f(x)$ to include $x = 3$.

Exercise 3

Give the formula for the extended function of

$$y = \frac{x^2 - 9}{x + 3}$$

that is continuous at $x = -3$.

Continuous Functions

- A function is **continuous on an interval** if and only if it is continuous on every point on the interval
- A **continuous function** is one that is continuous on every point of its domain.
  - A continuous function need not be continuous on every interval
  - $f(x) = \frac{1}{x}$ is continuous, because $x = 0$ is not part of its domain.
  - $f(x) = \frac{1}{x}$ is not continuous on the interval $[-1, 1]$

Continuous Functions

- Polynomial functions are continuous at every real number $x = c$.
- Rational functions are continuous at every point of their domains.
  - They have points of discontinuity at the zeros of their denominators.
- Absolute value functions $f(x) = |x|$ are continuous at every real number $x = c$.

Continuous Functions

- Exponential functions $f(x) = ae^{bx}$ are continuous at every real number $x = c$.
- Logarithmic functions $f(x) = \log_b x$ are continuous at every real number $x > 0$.
- Radical functions $f(x) = \sqrt[n]{x}$ are continuous at every real number that is an element of their domain.

Exercise 4

Explain why

$$f(x) = \frac{1}{x-3}$$

is continuous.

Algebraic Combinations

Algebraic combinations of continuous functions are continuous wherever they are defined.

Properties of Continuous Functions

If the functions f and g are continuous at $x = c$, then the following combinations are continuous at $x = c$.

1. Sums: $f + g$
2. Differences: $f - g$
3. Products: $f \cdot g$
4. Constant multiples: $k \cdot f$ $\forall k$
5. Quotients: f / g $g(c) \neq 0$

Composites

All composites of continuous functions are continuous. For example,

$$y = \sin(x^2)$$

is continuous.

$f(x) = \sin x$
$g(x) = x^2$
$f \circ g(x) = f(g(x))$
$= \sin(g(x))$
$= \sin(x^2)$

Composites of Continuous Functions

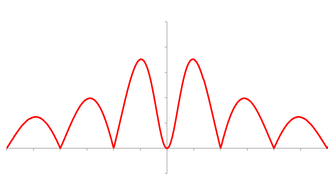
If f is continuous at c and g is continuous at $f(c)$, the composite $g \circ f$ is continuous at c.

Example 5

Show that

$$y = \left|\frac{x \sin x}{x^2 + 2}\right|$$

is continuous.



Let

$$g(x) = |x|$$
$$f(x) = \frac{x \sin x}{x^2 + 2}$$
$$y = g \circ f$$

$g(x)$ is continuous.
$f(x)$ is continuous. (algebraic combinations)
y is continuous. (composite of continuous functions)

Exercise 5

Show that

$$f(x) = \sqrt{\frac{x}{x+1}}$$

is cintinuous.

Intermediate Value Property

A function has the intermediate value property if it never takes on two values without taking on every value in between.

The Intermediate Value Theorem

- If f is continuous on the closed interval $[a, b]$ and k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that $f(c) = k$.
- One application of the Intermediate Value Theorem is to show that a function has a root in a given closed interval.

Example 6

Is any real number exactly one less than its cube?

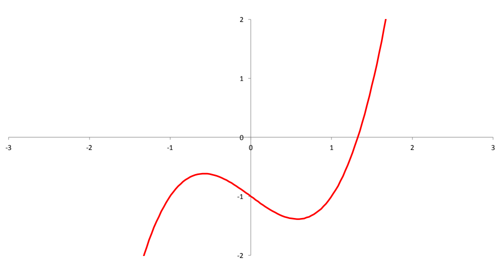
$$x = x^3 - 1$$
$$x^3 - x - 1 = 0$$

Let

$$f(x) = x^3 - x - 1$$

Does $f(x)$ have any zeros?

Let's graph it.



The curve crosses the x-axis between 1 and 2. By the intermediate value theorem, there must be a point c between 1 and 2 where $f(c) = 0$.

Exercise 6

Is any real number exactly two more than its cube?

If so, give its value accurate to 3 decimal places.

Homework

P 84: 3-30, multiples of 3, 42, 45, 51, 63