Chapter 2.2

Limits Involving Infinity

Objectives

- Finite Limits as $x \to \pm\infty$
- Sandwich Theorem Revisited
- Infinite Limits as $x \to a$
- End Behavior Models
- "Seeing" Limits as $x \to \pm\infty$

Learning Target

80% of the students will be able to find the following limit:

$$\lim_{x \to \infty} \sin \frac{1}{x}$$

Standard

G-GMD.1 Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone. *Use dissection arguments, Cavalieri's principle, and informal limit arguments.*

Overview

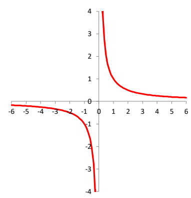
- Limits can be used to describe the behavior of functions when the absolute value of their argument becomes very large.
- We will learn how to functions under these conditions.
- Functions often have finite limits as $x \to \pm\infty$.

Infinity

- The symbol for infinity (∞) does not represent a real number.
- $\lim_{x\to\infty} f(x)$ describes the behavior of the function as x continues move far to the right on the number line.
- $\lim_{x\to-\infty} f(x)$ describes the behavior of the function as x continues move far to the left on the number line.

Horizontal Asymptote

- Consider $f(x) = \frac{1}{x}$
- $\lim_{x \to \infty} \frac{1}{x} = 0$
- $\lim_{x \to -\infty} \frac{1}{x} = 0$
- $y = 0$ is a horizontal asymptote



Horizontal Asymptote

The line $y = b$ is a horizontal asymptote of the graph of a function $y = f(x)$ if either

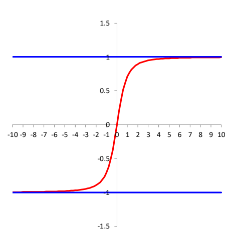
$$\lim_{x \to \infty} f(x) = b$$

or

$$\lim_{x \to -\infty} f(x) = b$$

Example 1

- Find all horizontal asymptotes of $f(x) = \frac{x}{\sqrt{x^2+1}}$
- $\lim_{x \to \infty} \frac{x}{\sqrt{x^2+1}} = 1$
- $\lim_{x \to -\infty} \frac{x}{\sqrt{x^2+1}} = -1$



Example 1

Let's calculate some numbers

X	Y_1
0	0
.5	.44721
1	.70711
1.5	.83205
2	.89443
2.5	.92848
3	.94868

X=0

X	Y_1
0	0
1	.70711
2	.89443
3	.94868
4	.97014
5	.98058
6	.98639

X=0

X	Y_1
0	0
2	.89443
4	.97014
6	.98639
8	.99228
10	.99504
12	.99655

X=0

X	Y_1
0	0
-.5	-.4472
-1	-.7071
-1.5	-.8321
-2	-.8944
-2.5	-.9285
-3	-.9487

X=0

X	Y_1
0	0
-1	-.7071
-2	-.8944
-3	-.9487
-4	-.9701
-5	-.9806
-6	-.9864

X=0

X	Y_1
0	0
-2	-.8944
-4	-.9701
-6	-.9864
-8	-.9923
-10	-.995
-12	-.9965

X=0

Exercise 1

Use graphs and tables to find

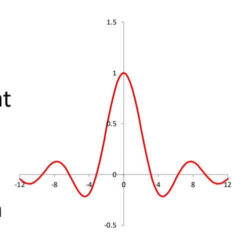
a. $\lim_{x\to\infty} \frac{3x+1}{|x|+2}$

b. $\lim_{x\to-\infty} \frac{3x+1}{|x|+2}$

Identify the horizontal asymptotes of $f(x) = \frac{3x+1}{|x|+2}$.

Example 2

- Find $\lim_{x\to\infty} \frac{\sin x}{x}$
- The graph suggests that $\lim_{x\to\infty} \frac{\sin x}{x} = 0$
- The sandwich theorem holds as $x \to \pm\infty$



Example 2

- $-1 \leq \sin x \leq 1$
- $-\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$ since $x > 0$
- $0 = \lim_{x\to\infty}\left(-\frac{1}{x}\right) = \lim_{x\to\infty}\frac{\sin x}{x} = \lim_{x\to\infty}\frac{1}{x} = 0$
- Since $\frac{\sin x}{x}$ is an even function, $\lim_{x\to-\infty}\frac{\sin x}{x} = 0$

Exercise 2

- Find $\lim_{x\to\infty}\frac{1-\cos x}{x}$

Properties of Limits as $x \to \pm\infty$

$\forall\, L, M, k \in \mathbb{R}$ and

$\lim_{x\to\pm\infty} f(x) = L$ and $\lim_{x\to\pm\infty} g(x) = M$

1. Sum Rule: $\lim_{x\to\pm\infty}\big(f(x) + g(x)\big) = L + M$
2. Difference Rule: $\lim_{x\to\pm\infty}\big(f(x) - g(x)\big) = L - M$
3. Product Rule: $\lim_{x\to\pm\infty}\big(f(x) \cdot g(x)\big) = L \cdot M$

Properties of Limits as $x \to \pm\infty$

$\forall\, L, M, k \in \mathbb{R}$ and

$$\lim_{x\to\pm\infty} f(x) = L \quad \text{and} \quad \lim_{x\to\pm\infty} g(x) = M$$

4. Constant Multiple Rule: $\lim_{x\to\pm\infty}\big(k\cdot f(x)\big) = k\cdot L$
5. Quotient Rule: $\lim_{x\to\pm\infty}\left(\frac{f(x)}{g(x)}\right) = \frac{L}{M}, M \neq 0$
6. Power Rule: $\lim_{x\to\pm\infty}\left(\big(f(x)\big)^{r/s}\right) = L^{r/s}$

   $\forall r, s \in \mathbb{Z}, s \neq 0$

Example 3

Find $\lim_{x\to\pm\infty}\left(\frac{5x+\sin x}{x}\right)$

$$\frac{5x+\sin x}{x} = \frac{5x}{x} + \frac{\sin x}{x} = 5 + \frac{\sin x}{x}$$

$$\lim_{x\to\pm\infty}\left(\frac{5x+\sin x}{x}\right) = \lim_{x\to\pm\infty}(5) + \lim_{x\to\pm\infty}\left(\frac{\sin x}{x}\right)$$

$$= 5 + 0 = 5$$

Exercise 3

Find

a. $\lim_{x\to\infty}\left(\frac{\sin x}{2x^2+x}\right)$

b. $\lim_{x\to-\infty}\left(\frac{\sin x}{2x^2+x}\right)$

Exploring Limits as $x \to \pm\infty$

1. Let $f(x) = 5x + \sin x$ and $g(x) = x$
   Do the limits of f and g exist as $x \to \infty$?
   Does the limit of the quotient exist?
2. Let $f(x) = \sin^2 x$ and $g(x) = \cos^2 x$
   Describe the behavior of f and g as $x \to \infty$
   Can we apply the sum rule to
   $\lim_{x\to\infty} \big(f(x) + g(x)\big)$?
   Does the limit of the sum exist?

Exploring Limits as $x \to \pm\infty$

3. Let $f(x) = \ln(2x)$ and $g(x) = \ln(x+1)$
   Find the limits of f and g as $x \to \infty$.
   Can we apply the difference rule to
   $\lim_{x\to\infty} \big(f(x) - g(x)\big)$?
   Does the limit of the difference exist?
4. Based on parts 1–3, what advice might you give about applying the properties of Limits as $x \to \pm\infty$?

Vertical Asymptote

The line $x = a$ is a vertical asymptote if either

$$\lim_{x\to a^+} f(x) = \pm\infty$$

or

$$\lim_{x\to a^-} f(x) = \pm\infty$$

Example 4

- Find the vertical asymptotes of $f(x) = \frac{1}{x^2}$. Describe the behavior to the left and right of each vertical asymptote.
- The values of the function approach ∞ on either side of $x = 0$.

$$\lim_{x \to 0^+} \frac{1}{x^2} = \infty \quad \text{and} \quad \lim_{x \to 0^-} \frac{1}{x^2} = \infty$$

- The line $x = 0$ is the only vertical asymptote.

Exercise 4

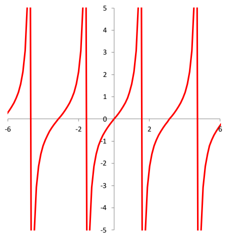
Find the vertical asymptotes of $f(x) = \frac{1}{x^2-4}$. Describe the behavior to the left and right of each vertical asymptote.

Finding Vertical Asymptotes

- $\lim_{x \to 0} \frac{1}{x^2} = \infty$
- $\lim_{x \to 0} \frac{1}{x}$ is undefined.
- Why?

Example 5
Finding Vertical Asymptotes

- $f(x) = \tan x = \frac{\sin x}{\cos x}$
- Infinitely many asymptotes.
- $\lim_{x \to \frac{(2n-1)\pi}{2}^+} \tan x = -\infty$
- $\lim_{x \to \frac{(2n-1)\pi}{2}^-} \tan x = \infty$



Exercise 5

Find the vertical asymptotes of $f(x) = \cot x$. Describe the behavior to the left and right of each vertical asymptote.

Denominator Equals 0

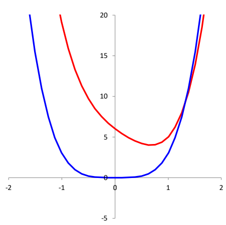
- Does every function have a vertical asymptote whenever the denominator is 0?
- $\lim_{x \to 0} \frac{\sin x}{x} = 1$

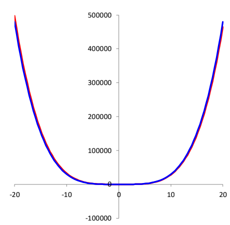
End Behavior Models

We can sometimes model the behavior of a complicated function with a simpler one that behaves in virtually the same way.

Example 6

- $f(x) = 3x^4 - 2x^3 + 3x^2 - 5x + 6$
- $g(x) = 3x^4$





Example 6

- $\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = \lim_{x\to\pm\infty} \left(\frac{3x^4-2x^3+3x^2-5x+6}{3x^4}\right)$
- $= \lim_{x\to\pm\infty} \left(1 - \frac{2}{3x} + \frac{1}{x^2} - \frac{5}{3x^3} + \frac{2}{x^4}\right)$
- $= 1$

Exercise 6

Find a power function end behavior model for f. Identify any horizontal asymptotes.

a. $f(x) = -4x^3 + x^2 - 2x - 1$

b. $f(x) = \frac{x-2}{2x^2+3x-5}$

End Behavior Model

The function g is

a. a **right end behavior model** for f if and only if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$

b. a **left end behavior model** for f if and only if $\lim_{x\to-\infty} \frac{f(x)}{g(x)} = 1$

End Behavior Model

- If one function provides both a left and right end behavior model, it is simply called an **end behavior model**.
- The end behavior of all polynomial functions behaves like the end behavior of monomials.

Example 7

Find an end behavior model for

$$f(x) = \frac{2x^5 + x^4 - x^2 + 1}{3x^2 - 5x + 7}$$

Find end behavior models for both the numerator and denominator.

$$f(x) = \frac{2x^5}{3x^2} = \frac{2}{3}x^3$$

Example 7

Find an end behavior model for

$$f(x) = \frac{2x^3 - x^2 + x - 1}{5x^3 + x^2 + x - 5}$$

Find end behavior models for both the numerator and denominator.

$$f(x) = \frac{2x^3}{5x^3} = \frac{2}{5}$$

$y = \frac{2}{5}$ is a horizontal asymptote.

Exercise 7

Find a power function end behavior model for

$$f(x) = \frac{4x^3 - 2x + 1}{x - 2}$$

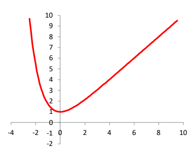
Identify any horizontal asymptotes.

Power Function

- A rational function always has a power function as an end behavior model.
- A function's left and right end behavior models may be different.

Example 8

- $f(x) = x + e^{-x}$



- $\lim_{x \to \infty} \frac{x+e^{-x}}{x} = \lim_{x \to \infty} \left(1 + \frac{e^{-x}}{x}\right) = 1$
- $\lim_{x \to -\infty} \frac{x+e^{-x}}{e^{-x}} = \lim_{x \to -\infty} \left(\frac{1}{e^{-x}} + 1\right) = 1$

Exercise 8

Find simple, basic functions as right end behavior and left end behavior models for,

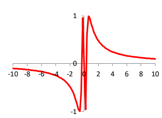
$$f(x) = x^2 + e^{-x}$$

“Seeing” Limits as $x \to \pm\infty$

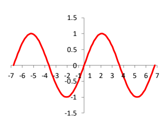
- $\lim_{x\to\infty}(x) = \lim_{x\to 0}\left(\frac{1}{x}\right)$
- Compare graphs
  - $y = f(x)$ as $x \to \pm\infty$
  - $y = f\left(\frac{1}{x}\right)$ as $x \to 0$

Example 9
Using Substitution

- Find $\lim_{x\to\infty}\left(\sin\left(\frac{1}{x}\right)\right)$
- Graph $y = \sin\left(\frac{1}{x}\right)$



- Graph $y = \sin(x)$



- $\lim_{x\to\infty}\left(\sin\left(\frac{1}{x}\right)\right) = 0$

Exercise 9

- Use a graph of $y = f\left(\frac{1}{x}\right)$ to find
  a. $f(x) = \lim_{x\to\infty}(xe^{x})$
  b. $f(x) = \lim_{x\to-\infty}(xe^{x})$

Homework

P 76: 3-54, multiples of 3, 59, 68, 70