Chapter 2.1

Limits and Continuity

Objectives

- Average and Instantaneous Speed
- Definition of Limit
- Properties of Limits
- One-sided and Two-sided Limits
- Sandwich Theorem

Learning Target

80% of the students will be able to find the following limit:

$$\lim_{x \to 0} \frac{x}{\sqrt{x+1} - 1}$$

Standard

G-GMD.1 Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone. *Use dissection arguments, Cavalieri's principle, and informal limit arguments.*

Overview

- Limits are the foundation of calculus, and they distinguish calculus from algebra and trigonometry.
- We will learn how to define and calculate limits.
- Limits can be used to test functions for continuity.

Average Speed

- Average speed of a particle in motion:
  - Find the change in position.
  - Find the elapsed time.
  - Divide the change in position by the elapsed time.

Definition

- If a particle moves from the position x_1 at time t_1 to the position x_2 at time t_2, the changes in position and time are,
  - $\Delta x = x_2 - x_1$
  - $\Delta t = t_2 - t_1$
- The average speed over this interval is,
  - $\frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1}$

Example 1: Finding an Average Speed

- A rock falls from the top of a cliff. What is its average speed during the first 2 seconds of its fall?
- For a dropped object,
  - $y = 16t^2$
  - $\frac{\Delta y}{\Delta t} = \frac{16(2)^2 - 16(0)^2}{2-0} = 32\ {}^{ft}/_{s}.$

Exercise 1

- An object dropped from rest from a tall building falls $y = 9.8t^2$ meters in the first t seconds. Find the average speed during the first 2 seconds.

Example 1: Finding an Instantaneous Speed

- Find the speed of the rock in Example 1 at the instant $t = 2$.
- Calculate the average speed of the rock from $t = 2$ to a slightly later time $t = 2 + h$.
  - $\frac{\Delta y}{\Delta t} = \frac{16(2+h)^2 - 16(2)^2}{h}$.
- Since we cannot divide by 0, we cannot evaluate this fraction directly.

Example 2. continued

- Let's try some algebraic manipulations.
  - $\frac{\Delta y}{\Delta t} = \frac{16(2+h)^2 - 16(2)^2}{h}$
  - $\frac{\Delta y}{\Delta t} = \frac{16(4+4h+h^2) - 16 \cdot 4}{h}$
  - $\frac{\Delta y}{\Delta t} = \frac{64+64h+16h^2-64}{h}$
  - $\frac{\Delta y}{\Delta t} = \frac{64h+16h^2}{h} = 64 + 16h$
  - $\frac{\Delta y}{\Delta t} = 64$

Exercise 2

- An object dropped from rest from a tall building falls $y = 9.8t^2$ meters in the first t seconds. Find the instantaneous speed the speed of the object at t = 2 seconds.

Limit

- Numerical limits of the ***value of a function***.
- Language describing how functions behave.
- Confirm algebraically.
- Cannot always confirm algebraically.
  - $f(x) = \frac{\sin x}{x}$

Definition: Limit

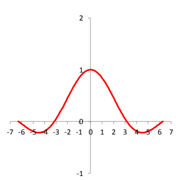
Assume the function f is defined in the neighborhood of c, and let c and L be real numbers. The function **f has limit L as x approaches c** if, given any positive number ε there is a positive number δ such that for all x,

$$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

$$\lim_{x \to c} f(x) = L$$

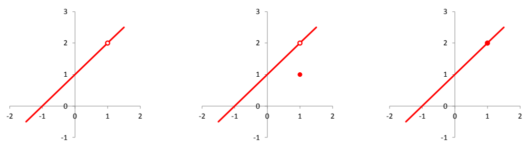
Limits

- $\lim_{h \to 0} \frac{16(2+h)^2 - 16(2)^2}{h} = 64$
- $\lim_{x \to 0} \frac{\sin x}{x} = 1$



Limit vs. Function Value

$f(x) = \frac{x^2 - 1}{x - 1}$ $\quad g(x) = \begin{cases} \frac{x^2 - 1}{x - 1}, x \neq 1 \\ 1, x = 1 \end{cases}$ $\quad h(x) = x + 1$



$$\lim_{x \to 1} f(x) = \lim_{x \to 1} g(x) = \lim_{x \to 1} h(x) = 2$$

Properties of Limits

$\forall\, L, M, c, k \in \mathbb{R}$, $\lim_{x \to c} f(x) = L$, $\lim_{x \to c} g(x) = M$

1. Sum Rule: $\lim_{x \to c}\big(f(x) + g(x)\big) = L + M$
2. Difference Rule: $\lim_{x \to c}\big(f(x) - g(x)\big) = L - M$
3. Product Rule: $\lim_{x \to c}\big(f(x) \cdot g(x)\big) = L \cdot M$

Properties of Limits

$\forall\, L, M, c, k \in \mathbb{R}$, $\lim_{x \to c} f(x) = L$, $\lim_{x \to c} g(x) = M$

4. Constant Multiple Rule: $\lim_{x \to c}\big(k \cdot f(x)\big) = k \cdot L$
5. Quotient Rule: $\lim_{x \to c}\left(\frac{f(x)}{g(x)}\right) = \frac{L}{M}, M \neq 0$
6. Power Rule: $\lim_{x \to c}\left(\big(f(x)\big)^{r/s}\right) = L^{r/s}$ $\quad \forall r, s \in \mathbb{Z}, s \neq 0$

Example 3

1. Find $\lim_{x \to c}(x^3 + 4x^2 - 3)$
   $= \lim_{x \to c}(x^3) + \lim_{x \to c}(4x^2) - \lim_{x \to c}(3)$
   $= c^3 + 4c^2 - 3$
2. Find $\lim_{x \to c}\left(\frac{x^4+x^2-1}{x^2+5}\right)$
   $= \frac{\lim_{x \to c}(x^4) + \lim_{x \to c}(x^2) - \lim_{x \to c}(1)}{\lim_{x \to c}(x^2) + \lim_{x \to c}(5)}$
   $= \frac{c^4 + c^2 - 1}{c^2 + 5}$

Exercise 3

1. Find $\lim_{x \to c} f(3x^3 - 2x^2 - 3)$
2. Find $\lim_{x \to c} f\left(\frac{x^5-2x^2+1}{x^3-5}\right)$

Polynomial and Rational Functions

1. If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$
   Is any polynomial function, $\forall c \in \mathbb{R}$,
   $\lim_{x \to c} f(x) = f(c) = a_n c + a_{n-1}c^{n-1} + \cdots + a_1 c + a_0$
2. If $f(x)$ and $g(x)$
   are polynomial functions, $\forall c \in \mathbb{R}$,
   $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f(c)}{g(c)}, g(c) \neq 0$

Example 4

1. Find $\lim_{x \to 3}[x^2(2-x)]$
$= (3)^2(2-3)$
$= -9$
2. Find $\lim_{x \to 2} \frac{x^2+2x+4}{x+2}$
$= \frac{(2)^2 + 2 \cdot 2 + 4}{2+2}$
$= \frac{4+4+4}{2+2} = \frac{12}{4} = 3$

Exercise 4

1. Find $\lim_{x \to 3} f(3x^3 - 2x^2 - 3)$
2. Find $\lim_{x \to 3} f\left(\frac{x^5-2x^2+1}{x^3-5}\right)$

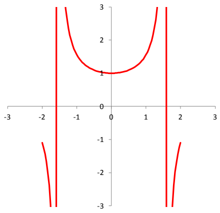
Limits of Other Functions

- The limits of many functions can be found by substitution at points where they are defined.
  - Trigonometric functions
  - Exponential functions
  - Logarithmic functions
  - Composites

Example 5

Determine $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)$

The graph suggests that the limit exists and that $\lim_{x\to 0}\left(\frac{\tan x}{x}\right) = 1$

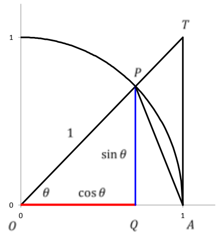


Example 5, continued

- $\lim_{x\to 0}\left(\frac{\tan x}{x}\right) = \lim_{x\to 0}\left(\frac{\sin x}{\cos x}\cdot\frac{1}{x}\right)$
- $= \lim_{x\to 0}\left(\frac{1}{\cos x}\cdot\frac{\sin x}{x}\right)$
- $= \lim_{x\to 0}\left(\frac{1}{\cos x}\right)\lim_{x\to 0}\left(\frac{\sin x}{x}\right)$
- $= 1\cdot\lim_{x\to 0}\left(\frac{\sin x}{x}\right)$
- $= \lim_{x\to 0}\left(\frac{\sin x}{x}\right)$

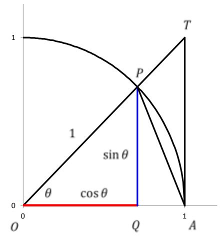
Example 5, continued

- The area of the inscribed triangle ΔOAP is,
  $$\frac{1}{2}\sin\theta$$
- The area of the sector OAP is,
  $$\frac{1}{2}\theta$$
- The area of the triangle ΔOAT is,
  $$\frac{1}{2}\tan\theta$$



Example 5, continued

- $\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta$
- $\sin\theta < \theta < \tan\theta$
- $1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$
- $\cos\theta < \frac{\sin\theta}{\theta} < 1$
- $\lim_{x\to 0}\left(\frac{\sin x}{x}\right) = 1$
- $\lim_{x\to 0}\left(\frac{\tan x}{x}\right) = 1$

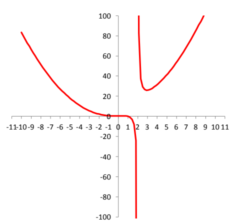


Exercise 5

Find $\lim_{x\to 0}\left(\frac{\sin^2 x}{x}\right)$

Example 6

- Sometimes a graph can show that a limit does not exist.
- $\lim_{x\to 2}\left(\frac{x^3-1}{x-2}\right)$



Exercise 6

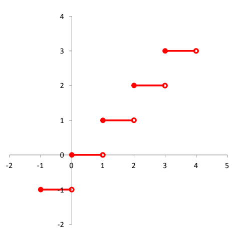
Find $\lim_{x \to 1} \left(\frac{x^4 - 4}{x - 1} \right)$

One-sided Limits

- Sometimes the limit $\lim_{x \to c} f(x)$ is different if we approach c from opposite sides.
- If we approach c from the right, we have the
  - right-hand limit: $\lim_{x \to c^+} f(x)$
- If we approach c from the left, we have the
  - left-hand limit: $\lim_{x \to c^-} f(x)$

Example 7

- Find the right-hand and left-hand limits of $f(x) = \text{int}\, x$



- $\lim_{x \to 3^+} \text{int}\, x = 3$
- $\lim_{x \to 3^-} \text{int}\, x = 2$

Exercise 7

Find $\lim_{x \to 0^+} \operatorname{int} x$

Find $\lim_{x \to 0^-} \operatorname{int} x$

Two Sided Limit

- If the left hand limit and the right-hand limits exist and are equal, then the limit is ***two-sided***.
- For a limit to exist, it must be two-sided.

Two-Sided Limits

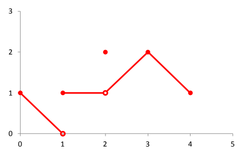
A function $f(x)$ has a limit as x approaches c if and only if the left-hand and right-hand limits exist at c, and they are equal.

$$\lim_{x \to c} f(x) = L \Leftrightarrow$$

$$\lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L$$

Example 8

$$f(x) = \begin{cases} -x+1, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \\ 2, & x = 2 \\ x-1, & 2 < x \le 3 \\ -x+5 & 3 < x \le 4 \end{cases}$$



Example 8

- At $x = 0$ $\lim_{x \to 0^+} f(x) = 1.$
- At $x = 1$ $\lim_{x \to 1^-} f(x) = 0.$
  $\lim_{x \to 1^+} f(x) = 1.$
  $\lim_{x \to 1} f(x)$ undefined.

Example 8

- At $x = 2$ $\lim_{x \to 2^-} f(x) = 1.$
  $\lim_{x \to 2^+} f(x) = 1.$
  $\lim_{x \to 2} f(x) = 1.$
  $f(2) = 2.$

Example 8

- At $x = 3$ $\lim_{x\to 3^-} f(x) = 2.$

  $\lim_{x\to 3^+} f(x) = 2.$

  $\lim_{x\to 3} f(x) = 2.$

  $f(3) = 2.$

- At $x = 4$ $\lim_{x\to 4^-} f(x) = 1.$

Exercise 8

Find

a. $\lim_{x\to -1^+} f(x)$

b. $\lim_{x\to 0^-} f(x)$

c. $\lim_{x\to 0^+} f(x)$

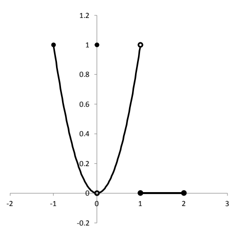
d. $\lim_{x\to 0} f(x)$

e. $\lim_{x\to 1^-} f(x)$

f. $\lim_{x\to 1^+} f(x)$

g. $\lim_{x\to 1} f(x)$

h. $\lim_{x\to 2} f(x)$



Sandwich Theorem

If we cannot find a limit directly, we may be able to find it using the sandwich theorem.

If $g(x) \le f(x) \le h(x)$ for all $x \ne c$ for some interval about c, and

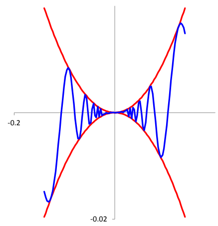
$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L\ ,$$

then

$$\lim_{x\to c} f(x) = L\ .$$

Example 9

- Find $\lim_{x \to 0}[x^2 \sin(^1/_x)]$
- $\left|x^2 \sin\frac{1}{x}\right| = |x^2| \cdot \left|\sin\frac{1}{x}\right|$
- $\leq |x^2| \cdot 1 = x^2$
- $-x^2 \leq x^2 \sin\frac{1}{x} \leq x^2$
- $\lim_{x \to 0}(-x^2) = \lim_{x \to 0}(x^2) = 0$
- $\lim_{x \to 0}[x^2 \sin(^1/_x)] = 0$



Homework

- Page 66:
- 3-36 multiples of 3 (3,6,9,etc.),
- 38, 47, 50, 51, 54, 55, 61,64