Chapter 1-6

Trigonometric Functions

Objectives

- convert between radians and degrees, and find arc length
- identify the periodicity and even-odd properties of trigonometric functions
- find the values of trigonometric functions
- generate the graphs of trigonometric functions and explore various transformations upon these graphs
- use the inverse trigonometric functions to solve problems

Learning Target

80% of the students will be able to evaluate $\sin(35\pi/2)$ without using a calculator.

Standard

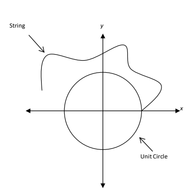
- F-IF.7e Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

Overview

- Radian Measure
- Graphs of Trigonometric Functions
- Periodicity
- Even and Odd Trigonometric Functions
- Transformations of Trigonometric Graphs
- Inverse Trigonometric Functions

Unit Circle

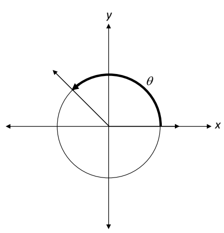
- circle centered on the origin
- radius of this circle is 1
- unit circle
- string attached to the unit circle
- stretch the string and wrap it counterclockwise



Winding Function

- input is the length of the string
- output is the coordinates of the point where the end of the string lies, (x, y)
- vertex lies on origin
- standard position
- terminal side passes through (x, y)

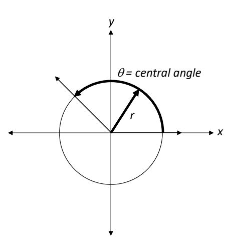
Radian Measure



length of the string is the radian measure of the angle

Arc Length

- Radian measure of central angle times radius.
- $s = r\theta$



Example 1

Find the length of an arc subtended on a circle of radius 3 by a central angle of measure $2\pi/3$.

$$s = r\theta = 3\frac{2\pi}{3} = 2\pi$$

Exercise 1

Find the length of an arc subtended on a circle of radius 5 by a central angle of measure $5\pi/4$.

Six Basic Trigonometric Functions

- angle of measure θ in standard position
- radius r

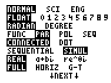
sine: $\sin\theta = \frac{y}{r}$ **cosecant:** $\csc\theta = \frac{r}{y}$

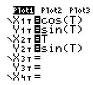
cosine: $\cos\theta = \frac{x}{r}$ **secant:** $\sec\theta = \frac{r}{x}$

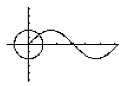
tangent: $\tan\theta = \frac{y}{x}$ **cotangent:** $\tan\theta = \frac{x}{y}$

Exploration

radian
parametric
simultaneous
square
$x_1 = \cos t$
$y_1 = \sin t$
$x_2 = t$
$y_2 = \sin t$







Exploration

- Use trace to compare the y-values of the two curves.
- Repeat with
  - $0 \le t \le 4\pi$
  - $[-1.5, 4\pi]$
  - $[-5, 5]$

Exploration

- $y_2 = \cos t$
- Use trace to compare the x-values of the two curves.

Exploration

- Repeat for
  - $y_2 = \tan t$
  - $y_2 = \csc t$
  - $y_2 = \sec t$
  - $y_2 = \cot t$
- with
  - $0 \leq t \leq 2\pi$
  - $[-1.5, 2\pi]$
  - $[-2.5, 2.5]$

Use trace to explore the curves.

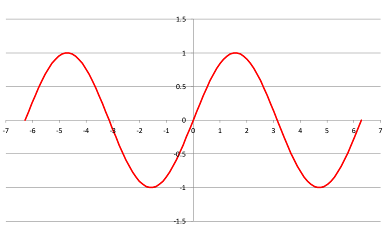
How is the y-value on curve 2 related to the corresponding point on curve 1?

Radian Measure

Unless otherwise stated, all angles will be measured in radians.

Keep your calculator in radian mode!

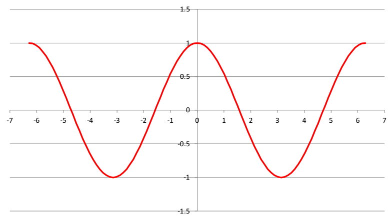
Sine



Domain: $-\infty < x < \infty$ Range: $-1 \leq y \leq 1$

Period: 2π

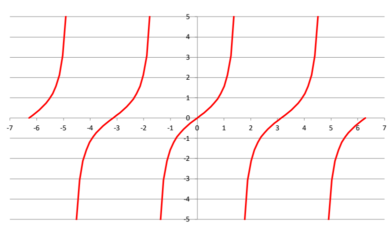
Cosine



Domain: $-\infty < x < \infty$ Range: $-1 \leq y \leq 1$

Period: 2π

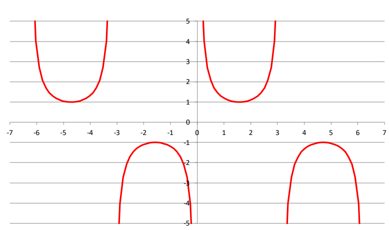
Tangent



Domain: $x \neq \pm\frac{2n-1}{2}\pi$ Range: $-\infty < y < \infty$

Period: π

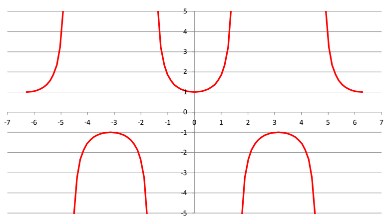
Cosecant



Domain: $x \neq 0, \pm n\pi$ Range: $y \leq -1, y \geq 1$

Period: 2π

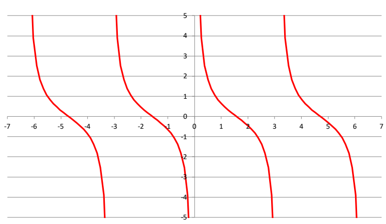
Secant



Domain: $x \neq \pm\frac{2n-1}{2}\pi$ Range: $y \leq -1, y \geq 1$

Period: π

Cotangent



Domain: $x \neq 0, \pm n\pi$ Range: $-\infty < y < \infty$

Period: π

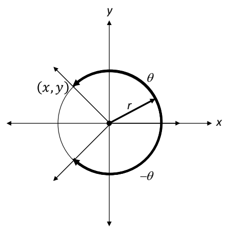
Periodicity

The six basic trigonometric functions are periodic

A function $f(x)$ is periodic if there is a positive number p such that $f(x+p) = f(x)\ \ \forall x$. The smallest such value of p is the period of f.

Example 2

Show that cosine is an even function and sine is odd.



$\cos\theta = \frac{x}{r} = \cos -\theta$

$\sin\theta = \frac{y}{r}$

$\sin(-\theta) = \frac{-y}{r}$

$\sin(-\theta) = -\sin\theta$

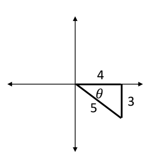
Exercise 2

Determine whether secant is odd or even.

Example 3

Find values of all the trigonometric functions of θ if

$\sin\theta = -\frac{3}{5}$ and $\tan\theta < 0$



$\csc\theta = -\frac{5}{3}$

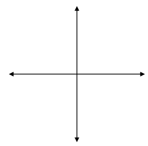
$\cos\theta = \frac{4}{5}$ and $\sec\theta = \frac{5}{4}$

$\tan\theta = -\frac{3}{4}$ and $\cot\theta = -\frac{4}{3}$

Exercise 3

Find values of all the trigonometric functions of θ if

$\cos\theta = -\frac{15}{17}$ and $\sin\theta < 0$



Transforming a Graph

$$y = af\big(b(x-c)\big) + \mathrm{d}$$

a vertical stretch or shrink
reflection about x-axis

b horizontal stretch or shrink
reflection about y-axis

c horizontal shift

d vertical shift

Sinusoid

$$f(x) = A\sin\left(\frac{2\pi}{B}(x-C)\right) + \mathrm{D}$$

$|A|$ amplitude

$|B|$ period

C horizontal shift

D vertical shift

Example 4

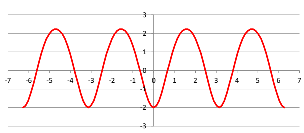
Determine the **(a)** period, **(b)** domain, **(c)** range, and **(d)** draw the graph of the function

$$y = 3\cos(2x - \pi) + 1$$

Rewrite the function,

$$y = 3\cos\left(2\left(x - \frac{\pi}{2}\right)\right) + 1$$

a. The period, B, is given by ${}^{2\pi}/_{B} = 2$
   The period is π
b. The domain is
   $(-\infty, \infty)$
c. The graph is a cosine with amplitude 3 that has been shifted up by 1 unit. The range is, $[-2,4]$
d.



Exercise 4

Determine the **(a)** period, **(b)** domain, **(c)** range, and **(d)** draw the graph of the function

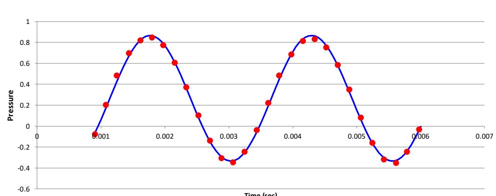
$$y = 2\sin\left(2x + \frac{\pi}{3}\right)$$

Example 5

The table on the next page depicts pressure vs. time data for the sound waves created by a tuning fork.

a. Find a sinusoidal regression equation and superimpose its graph on a scatter plot of the data.
b. The frequency of a wave is measured in Herz. (1Hz=1 cycle / second). Frequency = Period^{-1}

Time	Pressure	Time	Pressure	Time	Pressure
0.00091	-0.080	0.00271	-0.141	0.00453	0.749
0.00108	0.200	0.00289	-0.309	0.00471	0.581
0.00125	0.480	0.00307	-0.348	0.00489	0.346
0.00144	0.693	0.00325	-0.248	0.00507	0.077
0.00162	0.816	0.00344	-0.041	0.00525	-0.164
0.00180	0.844	0.00362	0.217	0.00543	-0.320
0.00198	0.771	0.00379	0.480	0.00562	-0.354
0.00216	0.603	0.00398	0.681	0.00579	-0.248
0.00234	0.368	0.00416	0.810	0.00598	-0.035
0.00253	0.099	0.00435	0.827		



- Sinusoidal regression equation
  $y = 0.6\sin(2488.6x - 2.832) + 0.266$
- Period
  $$\tau = \frac{2\pi}{2488.6}\ s$$
- Frequency
  $$\nu = \frac{2488.6}{2\pi} \approx 396\ Hz$$

Exercise 5

1. Perform a sinusoidal regression analysis on the data in this table.
2. Determine the frequency.

Time (ms)	Pressure	Time (ms)	Pressure
0.2368	1.29021	4.9024	-1.06632
1.0752	1.51411	5.6608	1.51411
1.8432	-0.89280	6.4288	1.43015
2.6016	-0.04758	7.1904	-1.51412
3.3728	1.51971	7.9584	1.46933
4.1312	0.32185	8.7168	1.50851

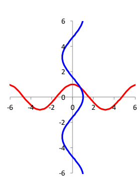
Inverse Trigonometric Functions

Trigonometric functions are periodic.

They are not one-to-one.

They do not have inverses.

Use parametric graphing.



Example 6

Restricted Domain of $\sin\theta$

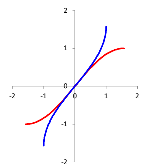
Restrict the domain of $\sin\theta$ to

$$-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

Use parametric graphing.

$\sin\theta$ is one-to-one,

It has an inverse.



Exercise 6

Show that $y = \tan\theta$ is one-to-one with domain $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.

Then graph its inverse.

Inverse of Sine

The inverse of the sine function is denoted,

$$\sin^{-1}\theta$$

It also called.

$$\arcsin\theta$$

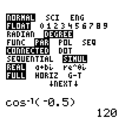
These are called “inverse sine of theta” or “arcsine of theta”.

Inverse Trigonometric Functions

Function	Domain	Range
$y = \sin^{-1} x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$y = \cos^{-1} x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = \tan^{-1} x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
$y = \sec^{-1} x$	$\lvert x \rvert \ge 1$	$0 \le y \le \pi, y \ne \frac{\pi}{2}$
$y = \csc^{-1} x$	$\lvert x \rvert \ge 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}, y \ne 0$
$y = \cot^{-1} x$	$-\infty < x < \infty$	$0 < y < \pi$

Example 7

Find the measure of $\cos^{-1}(-0.5)$ in degrees and radians. Find the exact answer.

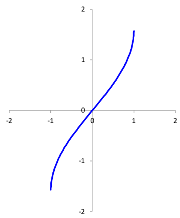


$$\frac{2\pi}{3} \approx 2.094395102$$

Exercise 7

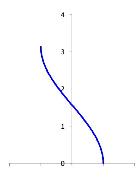
Find the measure of $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$ in degrees and radians. Find the exact answer.

Arcsine



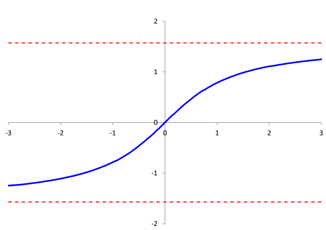
Domain: $-1 \leq x \leq 1$ Range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

Arccosine



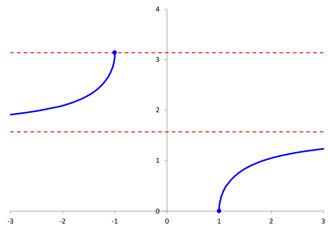
Domain: $-1 \leq y \leq 1$ Range: $0 \leq y \leq \pi$

Arctangent



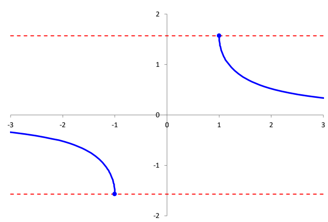
Domain: $-\infty < x < \infty$ Range: $\frac{\pi}{2} < y < \frac{\pi}{2}$

Arcsecant



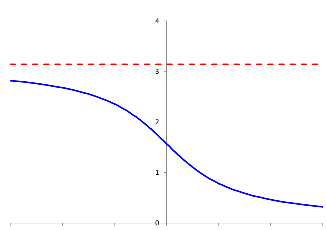
Domain: $x \leq -1$ or $x \geq 1$ Range: $0 \leq y \leq \pi, y \neq \frac{\pi}{2}$

Arccosecant



Domain: $x \leq -1$ or $x \geq 1$ Range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, y \neq 0$

Arccotangent



Domain: $-\infty < x < \infty$ Range: $0 < y < \pi$

Example 8

Solve for x.

a. $\sin x = 0.7,\ 0 \le x \le 2\pi$

$\sin^{-1}(0.7) \approx 0.775$

x is in 1st quadrant

$\sin(\pi - 0.775) = 0.7$

$x \approx 0.775, 2.366$

Example 8, continued

Solve for x.

b. $\tan x = -2,\ -\infty < x < \infty$

$\tan^{-1}(-2) \approx -1.107$

x is in 4th quadrant.

$x \approx -1.107 + k\pi$

Exercise 8

Solve for x.

$\cos x = -0.7,\ 2\pi \le x < 4\pi$

Homework

p. 52: #1-4 all, 5-8 all (draw graph to determine answer), 9, 10, 11-21 odds, 24 (plot and interpret the graph as an aid for answering the questions; compare with #43), 25-43 odds