Chapter 1-5

Functions and Logarithms

Objectives

- identify a one-to-one function
- determine the algebraic representation and the graphical representation of a function and its inverse
- use parametric equations to graph inverse functions
- apply the properties of logarithms
- use logarithmic regression equations to solve problems

Learning Target

80% of the students will be able to find the inverse of $f(x) = x^2 - 3$.

Standard

- F-IF.7e Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

Overview

- One-to-One Functions
- Inverses
- Finding Inverses
- Logarithmic Functions
- Properties of Logarithms
- Applications

One-to-One Function

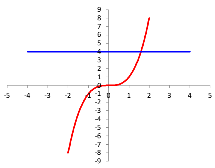
A function $f(x)$ is one-to-one on a domain D:

$$a \neq b \Rightarrow f(a) \neq f(b)$$

$$\forall a, b \in \mathbb{R}$$

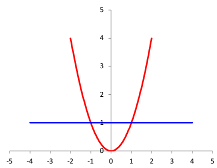
Horizontal Line Test

$y = x^3$



One-to-one: graph intersects each horizontal line once

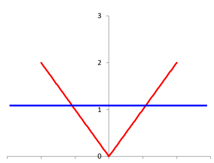
$y = x^2$



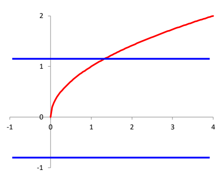
Not one-to-one: graph intersects some horizontal lines more than once

Example 1
Using the Horizontal Line Test

$f(x) = |x|$



$f(x) = \sqrt{x}$



Exercise 1

- Are the following functions one-to-one? Explain.
  - $y = 2|x|$
  - $y = \sqrt[3]{x}$

Inverses

- If a function is one-to-one, then it has an inverse.
- The inverse of a function undoes the action of a function.
- $f(f^{-1}(x)) = x$

Example of Inverses

Graph $f(x)$, $g(x)$, $f \circ g$, and $g \circ f$ for the following functions in a square window.

- $f(x) = x^3$, and $g(x) = x^{1/3}$
- $f(x) = x$, and $g(x) = \frac{1}{x}$
- $f(x) = 3x$, and $g(x) = \frac{x}{3}$
- $f(x) = e^x$, and $g(x) = \ln x$

Identity Function

Graph the following functions in a square window:

$f(x) = e^x, \qquad g(x) = \ln x, \qquad y = x$

Testing for Inverses

$$f \circ f^{-1} = x$$

$$f(x) = x^2 \qquad f^{-1}(x) = x^{1/2}$$

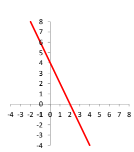
$$f(f^{-1}(x)) = f\left(x^{1/2}\right) = x$$

Finding Inverses

1. Write the function as $y = f(x)$.
2. Interchange x and y.
3. Solve for y.
4. The resulting function is $y = f^{-1}(x)$.

Example 2

Show that $y = -2x + 4$ is one-to-one and find its inverse.

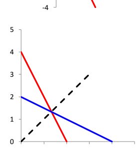


Interchange x and y.

$$x = -2y + 4$$

Solve for y.

$$y = -\frac{1}{2}x + 2$$



Exercise 2

Find f^{-1} and verify that

$$(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$$

- $f(x) = 5 - 4x$
- $f(x) = \frac{1}{x^3}$

Graphing Inverse Functions Parametrically

- Parametric graphing allows us to graph the inverse of a function $y = f(x)$.
- Set the parametric equations as follows:
  - $x_1 = t$
  - $y_1 = f(t)$
  - $x_2 = f(t)$
  - $y_2 = t$

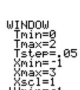
Example 3

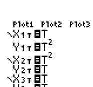
- Graph the one-to-one function

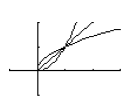
  $f(x) = x^2 \;\; x \geq 0$

  Together with its inverse and the line

  $y = x \;\; x \geq 0$







Example 3 continued

Find the inverse $f(x) = x^2$ as a function of x.

1. Interchange x and y.

   $x = y^2$

2. Solve for y.

   $y = x^{1/2} = \sqrt{x}$

Exercise 3

Use parametric graphing to graph f, f^{-1}, and $y = x$ where $f(x) = 3^{-x}$.

Logarithmic Functions

If a is any positive real number other than 1, then the exponential function with base a is one-to-one.

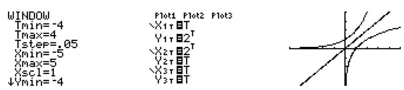
$$y = a^x$$

The base a logarithmic function is the inverse of the base a exponential function.

$$y = \log_a x$$
$$a > 0, a \neq 1$$

Parametric Graph

Use parametric graphing to graph
$y = 2^x$, $y = \log_2 x$, and $y = x$



Common and Natural Logarithms

- The common logarithm is the base 10 logarithm.
- Its primary use has been as an aid to multiplying and dividing.
- With the importance of powerful computing machines, its importance is diminishing.
- Natural logarithms are the base e logarithms.

Properties of Logarithms

$$a^{\log_a x} = x$$

$$\log_x a^x = x$$

$$e^{\ln x} = x$$

$$\ln e^x = x$$

Example 4

Solve for x

- $\ln x = 3t + 5$
  - $e^{\ln x} = e^{3t+5}$
  - $x = e^{3t+5}$
- $e^{2x} = 10$
  - $\ln e^{2x} = \ln 10$
  - $2x = \ln 10$
  - $x = \frac{1}{2}\ln 10 \approx 1.151$

Exercise 4

a. Solve for t $\quad 2^t = 5$

b. Solve for y $\quad \ln(y + 2) = x + \ln 2x$

Properties of Logarithms

For any real numbers $x > 0$ and $y > 0$

1. Product Rule: $\log_a xy = \log_a x + \log_a y$
2. Quotient Rule: $\log_a \frac{x}{y} = \log_a x - \log_a y$
3. Power Rule: $\log_a x^y = y \log_a x$

Exploration

Let $y_1 = \ln(ax)$, $y_2 = \ln x$, and $y_3 = y_1 - y_2$

1. Graph y_1 and y_2 for $a = 2, 3, 4,$ and 5.
2. Graph y_3 for $a = 2, 3, 4,$ and 5.
3. Find y_3 algebraically.

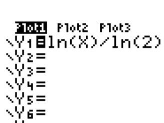
Change of Base

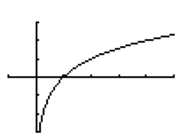
$$a^{\log_a x} = x$$
$$e^{\ln x} = x$$
$$a^{\log_a x} = e^{\ln x}$$
$$\ln\left(a^{\log_a x}\right) = \ln e^{\ln x} = \ln x$$
$$\log_a x \cdot \ln a = \ln x$$
$$\log_a x = \frac{\ln x}{\ln a}$$

Example 5

Graph $f(x) = \log_2 x$

$$f(x) = \frac{\ln x}{\ln 2}$$





Exercise 5

Graph $y = \log_5(x+2)$.

What are the domain and range?

Example 6

If you invest $24 in an account that pays 6% interest compounded annually, how long will it take until you have $48 billion?

$$24(1.06)^t = 4.8 \cdot 10^{10}$$
$$1.06^t = 2 \cdot 10^9$$
$$\ln(1.06^t) = \ln 2 \cdot 10^9$$
$$t \ln 1.06 = \ln 2 \cdot 10^9$$
$$t = \frac{\ln 2 \cdot 10^9}{\ln 1.06} \approx 367$$

Exercise 6

How much time would be required for $24 to increase to $48 billion if the interest rate is 5%?

Example 7
Logarithmic Regression Analysis

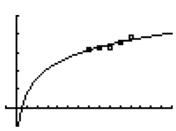
The following table depicts the natural gas production (in trillions of cubic feet per year) by Saudi Arabia for several years. Use a logarithmic regression analysis to predict the production in 2002.

Year	Production
1997	1.60
1998	1.65
1999	1.63
2000	1.76
2001	1.90

Example 7 continued

LnReg
y=a+blnx
a=.3729679138
b=.6111073717
r²=.785344423
r=.8861966051

Y₁(12)
1.891512686



The actual value was 2.00 trillion cubic feet.

Exercise 7

The following table depicts the natural gas production (in trillions of cubic feet per year) by Canada for several years. Use a logarithmic regression analysis to predict the production in 2002.

Compare your result to the actual production of 6.63 trillion cubic feet in 2002.

Year	Production
1997	5.76
1998	5.98
1999	6.26
2000	6.47
2001	6.60

Homework

p. 44: #1-12 all, 13-23 odds, 33-43 all,
45-48 all, 54-57 all